THE AMERICAN MATHEMATICAL MONTHLY

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association offers a new form of service to its members by the maintenance of an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office will not handle detailed recommendations, after the manner of a teachers' agency, but will supply certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim will be to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

TENTH ANNUAL MEETING OF THE MISSOURI SECTION

The tenth annual meeting of the Missouri Section of the Mathematical Association of America was held at the Junior College of Kansas City on November 13, 1926. Chairman G. E. Wahlin called the meeting to order at 9 A. M.

The attendance was twenty-one including the following eleven members of the Association:

E. F. Allen, B. Cosby, B. F. Finkel, R. R. Fleet, W. A. Luby, Arria Murto, A. D. Pierson, P. R. Rider, G. E. Wahlin, R. A. Wells, W. D. A. Westfall.

At a brief business session Professors Dunkel and Roever and the Secretary were appointed a committee to draft resolutions on the death of Professor C. A. Waldo, who was chairman of the committee which organized the Missouri Section and first chairman of the section. The following officers were elected for 1927: Chairman, KATHRYN WYANT, University of Missouri; Vice-Chairman, W. H. ROEVER, Washington University; Secretary-Treasurer, P. R. RIDER, Washington University.

The following papers were presented:

1. "Causes of the present popular attitude toward mathematics" by Professor R. A. WELLS, Park College.

2. "Desirable courses for students intending to do graduate work in mathematics" by Professor W. D. A. WESTFALL, University of Missouri.

3. "Unified courses in mathematics" by Professor BYRON INGOLD, Culver Stockton College.

4. "Some mathematical questions in Missouri history" by Miss KATHRYN WYANT, University of Missouri.

5. "Old and new concepts of mathematics" by Professor B. F. FINKEL, Drury College.

6. "Ten years of the Missouri section" by Professor P. R. RIDER, Washington University.

Abstracts of these papers are given below, the numbers corresponding to the numbers in the list of titles.

1. A study of the attitude of college students and of the educational public reveals the existence of much indifference and considerable antagonism to the study of mathematics. One frequently hears such expressions as, "I cannot learn mathematics," and "I do not care for mathematics." Attempts are continually being made to give it a less important place in the scheme of education. Professor Wells stated that the causes of this condition seem to be:

(a) Ignorance of the real nature of the subject matter of mathematics, caused usually by the ignorance or carelessness of teachers of elementary mathematics,

(b) The persistent agitation carried on by persons, willing to be known as educational experts, who have the public ear and who are continually trying to convince the public that there is no use in studying mathematics beyond the simple processes of arithmetic,

(c) Carelessness on the part of writers of text books on elementary mathematics shown in the use of inaccurate English and in the lack of regard for the technical use of mathematical terms.

The remedy seems to be for those who are interested in the subject of mathematics and who know what it really means to seize every opportunity to go before the educational public and give them some real information as to what mathematics is and why it should have a place in the scheme of education. Mathematics does not deserve a place in the curriculum because of its disciplinary value, or because of its practical application, or because of its training in the use of formal logic or in symbolic thinking, although these are all important reasons for studying it. But it does deserve a place in the scheme of education because of what it is. Mathematics is an important science with a perfectly definite subject matter and that subject matter is of vital interest to every human being. That is the reason why mathematics should have a place in the course of study. Teachers of mathematics should keep this idea before

the public and they should do it as persistently and insistently as the other group have worked in trying to discredit the study of mathematics.

2. The changing conditions in graduate schools that make greater homogeneity in the preparation of first year graduate students were considered by Professor Westfall. A tentative outline for a uniform undergraduate mathematical curriculum was proposed.

3. Professor Ingold's discussion involved three steps; first, setting forth arguments favorable to the giving of unified courses in undergraduate work; second, conditions unfavorable to such procedure; and third, an attempt at balancing these arguments with the final conclusion that courses be given separately under their respective headings. In the opinion of the author much depends upon the familiarity of the instructor with the correlated sciences and his ability to answer the eternal "why and wherefore" of the student concerning mathematical formulas of the sciences.

4. The first country newspaper to be published west of the Mississippi River was the *Missouri Intelligencer*. The paper was a weekly printed in Old Franklin, a little town on the Missouri River. Since a stage made a round trip between St. Louis and Franklin each week, bringing passengers and mail, the paper was able to keep the people of this newly settled territory in touch with the Eastern States.

It was in September, 1823, that one who signed himself "The Missouri Mathematician" sent several riddles and a problem in Diophantine Analysis to the Intelligencer. The problem was answered the next March by "A Subscriber from Howard County," who says he had "formerly, nearly half a century ago, taken great delight in studying mathematics." His solution, however, was not a general one but consisted of a set of numbers, each of which was greater than 10,000,000. The numbers check in the problem as "The Subscriber from Howard County" interpreted it, but it seems as if he had the wrong meaning of the problem. At the close of his discussion of this problem, "The Subscriber from Howard County" proposed two other questions, the first about the weight of an infinite cylinder the second another Diophantine problem.

Two months later "The Missouri Mathematican" answered these, but "for want of proper engraving" his solution to the Diophantine problem was never published. His answer to the question concerning the infinite cylinder was a harsh criticism of the proposer, stating that "an attempt to solve this question would appear like insanity." Two more questions were then proposed by "The Missouri Mathematician," one concerning the dimensions of a triangular field under certain conditions, the other a Diophantine problem concerning the dimensions of a certain trapezium.

In her paper, Miss Wyant gave a short history of Diophantine Analysis, a sketch of the local conditions in Missouri in 1823, and solutions to some of the problems proposed by the two men who, though interested in mathematics, seem to have left no trace of their identity other than "The Missouri Mathematician" and "The Subscriber from Howard County."

5. Professor Finkel's paper brought out a comparison of the old and new conceptions of such terms as number, mathematics, infinity, infinitesimal, derivative, etc. The paper dwelt at some length on the various conceptions of the postulates of Euclid, Saccheri, Lobatschewsky, Riemann, and the deductions drawn from them.

6. The Secretary sketched briefly the history of the organization of the Missouri Section and summarized the activities of the section during the ten years since its founding on November 18, 1916.

PAUL R. RIDER, *Secretary-Treasurer.*

DIOPHANTINE PROBLEMS IN WEIGHING

BY H. A. SIMMONS, Northwestern University

1. **Introduction.** The purpose of this paper is to solve certain systems of linear Diophantine equations by use of a lever and a system of n positive integral weights. A more precise statement of our problem will be given after we have made a few definitions and exhibited our method of procedure.

Definition 1. The symbol L_{jk}, j and k being any two positive integers, will stand for a lever whose left arm is j units in length and whose right arm is k units, there being places for hanging weights at distances of $1, 2, 3, \cdots, k$ units from the fulcrum on the right arm, and at distances of $1, 2, 3, \cdots, j$ units on the left, as shown in Fig. 1.

$$-j \cdots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \cdots k$$

$$\Lambda$$

FIG. 1

Definition 2. The symbol σ_n will stand for a system of n positive integral weights $w_1, w_2, w_3, \cdots, w_n$ where $w_1 \leqq w_2 \leqq w_3 \leqq \cdots \leqq w_n$; s_n will stand for the sum of these n weights.

Definition 3. Restricting the application of the word "object" throughout the paper to a thing whose weight in pounds can be expressed by a positive integer, we shall say that an object is "weighed" on L_{jk} if after it is placed at a division point $-p$ on the left arm of L_{jk}, where $1 \leqq p \leqq j$, the weights of σ_n are so placed at one or more division points q of L_{jk}, where $-j \leqq q \leqq k$, as to bring the lever into equilibrium. The symbols $1_0, 2_0, 3_0, \cdots$, wherever

they are used, will stand for objects of weights 1 lb., 2 lbs., 3 lbs., $\cdots$, which are to be weighed; the accented positive integers $1', 2', 3', \cdots$, wherever they are used, will stand for weights of 1 lb., 2 lbs., 3 lbs., $\cdots$, which belong to σ_n. These subscripts and accents will be omitted when the meaning intended is clear without them.

Definition 4. A system σ_n will be said to have a *consecutive weighing capacity* with respect to L_{jk} if it suffices to weigh on L_{jk} every object from 1_0 to $(ks_n)_0$ inclusive; otherwise σ_n will be called a non-consecutive system.

Example. If $j=k=1$ and $\sigma_n=\sigma_2$, the weights $w_1=1'$, $w_2=3'$ constitute a consecutive system σ_2 which suffices to weigh on L_{11} objects of all weights from 1_0 to $[k(w_1+w_2)]_0=4_0$ inclusive, but the weights $w_1=1'$, $w_2=4'$ constitute a non-consecutive system σ_2 which does not suffice to weigh 2_0.

Definition 5. Of two consecutive systems of n weights each, the system whose weights have the larger sum will be said to have the *greater consecutive capacity*.

With the above definitions holding, we proceed to find a system σ_n which the lever suggests as a possible consecutive system of large capacity; later we shall discuss completely the nature of the capacity of this system σ_n.

Since $w_1=1'$ is the only single weight which suffices to weigh 1_0 on L_{11} and since $w_1=1'$ suffices to weigh 1_0 on L_{jk}, whatever be the positive integral values of j, k, we select $w_1=1'$. When w_1 is placed at the points $1, 2, \cdots, k$ of L_{jk}, it suffices to weigh at -1 the objects $1_0, 2_0, \cdots, k_0$, respectively. The next object to be weighed is $(k+1)_0$. We place[1] it at -1, and we place $w_1=1'$ at $-j$. The moment thus created on the left arm of L_{jk} is $j'+(k+1)_0=j+k+1$. Hence we select $w_2=(k+j+1)'$. Now with w_2 at 1, we weigh w_0. We further observe that if w_2 is placed at p where p is a positive integer $\leq k$, w_1 can be used at the division points of L_{jk} to weigh at -1 objects of all weights from $pw_2-j=(p-1)w_2+k+1$ to pw_2+k inclusive. Therefore, by using all values of p from 1 to k, we weigh at -1 all objects from $1\cdot w_2-j=(k+1)_0$ to $kw_2+k=k(w_2+w_1)$. Adding these objects to those which w_1 alone suffices to weigh at -1, we form a consecutive set of objects running from 1_0 to $[k(w_1+w_2)]_0=[k[(j+k+1)^2-1]/(j+k)]_0$ inclusive. Furthermore, if we select $w_3=[(j+k+1)^2]'$ and carry one step further the method of procedure which we have just used, we find that w_1, w_2, w_3 suffice to weigh at -1 all objects from 1_0 to $(ks_3)_0=[k[(j+k+1)^3-1]/(j+k)]_0$ inclusive. These facts lead us to ask two questions:

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[1] That one cannot place $(k+1)_0$ at $-p$, $p>1$, and $w_1=1'$ at $-j$, and then select $w_2=j'+p(k+1)_0=j+pk+p$, for every pair of positive integers j, k is due to the fact, found in Theorem 4 of this paper, that if $k>1$ the weights $w_1=1'$ and $w_2=j+pk+p$ constitute a non-consecutive system σ_2.

(*A*) *If we select the system* σ_n *whose weights are*

$$w_1 = 1',\ w_2 = (j+k+1)',\ w_3 = [(j+k+1)^2]',\ \cdots,\ w_n = [(j+k+1)^{n-1}]', \tag{1}$$

with sum

$$s_n = \left[\frac{(j+k+1)^n - 1}{j+k}\right]', \tag{2}$$

can we weigh on L_{jk} *all integral weights from* 1_0 *to* $(ks_n)_0$ *inclusive?*

(*B*) *If the system* σ_n *of* (1) *is consecutive with respect to* L_{jk}, *does it have a greater weight sum* s_n, *given by* (2), *than has any other system of* n *weights which is consecutive with respect to* L_{jk}?

Through questions (A) and (B) we approach our main problem, which may be stated as follows. *Among all systems* σ_n *whose weighing capacities with respect to* L_{jk} *are consecutive* (*see Definition* 4), *to find that one whose weight sum is the greatest.*

Stated in Diophantine form, the problem is: *to find the set of* n *positive integers* $w_1, w_2, w_3, \cdots, w_n$ *of largest sum* s_n *for which there exists a consistent system of linear equations in* $w_1, w_2, w_3, \cdots, w_n$ *of the form*

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \qquad (i = 1,2,\cdots, ks_n), \tag{3}$$

where the a's and b's are integers satisfying the conditions

$$-j \leqq a_{ip} \leqq k, \qquad (p = 1,2,\cdots, n), \tag{4}$$

$$-j \leqq b_i \leqq -1, \tag{5}$$

j *and* k *being positive integers.*

Example. If $n=3$ and $j=k=1$, the largest s_3 is 13 and σ_3 consists of the weights $w_1=1'$, $w_2=3'$, $w_3=9'$. The 13 equations in (3) may then be written schematically as follows:

$$1\cdot 1+0\cdot 3+0\cdot 9- 1=0, \qquad -1\cdot 1+1\cdot 3+0\cdot 9- 2=0,$$
$$1\cdot 1+1\cdot 3+0\cdot 9- 4=0, \qquad -1\cdot 1-1\cdot 3+1\cdot 9- 5=0,$$
$$1\cdot 1-1\cdot 3+1\cdot 9- 7=0, \qquad -1\cdot 1+0\cdot 3+1\cdot 9- 8=0,$$
$$1\cdot 1+0\cdot 3+1\cdot 9-10=0, \qquad -1\cdot 1+1\cdot 3+1\cdot 9-11=0,$$
$$1\cdot 1+1\cdot 3+1\cdot 9-13=0,$$
$$0\cdot 1+1\cdot 3+0\cdot 9- 3=0,$$
$$0\cdot 1-1\cdot 3+1\cdot 9- 6=0,$$
$$0\cdot 1+0\cdot 3+1\cdot 9- 9=0,$$
$$0\cdot 1+1\cdot 3+1\cdot 9-12=0.$$

Insofar as is known to the author, this is the first time the above problem has been considered, though simple special cases of it have received some attention. We solve the above problem and certain modifications of it completely in this paper. In § 2, we give an inductive answer to question (A). In § 3, we develop a set of lemmas which are used in sections 4 to 8 inclusive to answer question (B). In § 9, we solve the problem which results when condition (4) above is replaced by

$$0 \leqq a_{ip} \leqq k, \qquad (p = 1,2, \cdots, n), \quad (6)$$

all other features of the above problem being unaltered. In § 10, we find the maximum consecutive capacity of a system of positive integral weights of which there are n types of weights $w_1, w_2, \cdots, w_n$ and m_r weights of type w_r $(r=1, 2, \cdots, n)$, with $m_r>1$ for at least one value of r, each weight being applicable as specified in condition (4). In § 11, we modify this problem merely by using condition (6) instead of (4) and we solve the resulting problem completely. For every problem which we treat, we obtain a unique solution, and do not avoid any case that arises, special or not. In § 12, we state the most general Diophantine problem that is solved in this paper, and then propose three problems to the reader.

Throughout the paper, we use the method of attack which is suggested by results of experiments with systems of weights σ_n and levers L_{jk}, though we admit that the Diophantine form of the problem is also interesting. The proofs of our more difficult theorems are made by use of *mathematical induction* together with a *set of lemmas*. In the lemmas certain inequalities develop which play a rôle in the proofs here somewhat like the rôle played by the inequalities used by Professor D. R. Curtiss[1] in solving the Kellogg Problem (*1922*, 380–387).

2. **Answer to (A).** We know from § 1 that the answer to this question is in the affirmative for $n=1, 2, 3$. We wish to prove that this is true for all positive integral values of n. Suppose it is true for $n=m$, a positive integer. Then from (1) the weights are

$$w_1 = 1', \quad w_2 = (j + k + 1)', \cdots, \quad w_m = [(j + k + 1)^{m-1}]', \qquad (7)$$

which suffice to weigh on L_{jk} objects of all integral weights from 1_0 to

$$(ks_m)_0 = \left[k \cdot \frac{(j + k + 1)^m - 1}{j + k}\right]_0 \qquad (8)$$

inclusive. The next object to be weighed is $w_0=(ks_m+1)_0$, and the next weight to be added to the set (7) is $w_{m+1}=[(j+k+1)^m]'=[(j+k)s_m+1]'$,

[1] The author is pleased to acknowledge his indebtedness to Professor Curtiss for his numerous helpful suggestions in regard both to the method of attack and to the form of this paper.

by (8). We observe now that if w_{m+1} is placed at p, where p is a positive integer $\leqq k$, the weights of (7) can be so placed at division points of L_{jk} as to weigh at -1 objects of all integral weights from $pw_{m+1}-js_m=(p-1)w_{m+1}+ks_m+1$ to $pw_{m+1}+ks_m$ inclusive. Therefore, by using all values of p from 1 to k, we weigh at -1 all integral weights from $1\cdot w_{m+1}-js_m=ks_m+1=w_0$ to $kw_{m+1}+ks_m=ks_{m+1}$. Adding these weights to those which the set (7) alone suffices to weigh at -1, we form a consecutive set of weights running from 1_0 to

$$(ks_{m+1})_0=\left[k\cdot\frac{(j+k+1)^{m+1}-1}{j+k}\right]_0 \tag{9}$$

inclusive. But (9) is the result which we obtain if in (8) we replace m by $m+1$. The induction is therefore complete, and the answer to (A) is in the affirmative for every positive integral value of n.

3. **Lemmas used in discussing (B).** *Definition* 6. The symbol Σ_n will stand for the system of weights which our problem, § 1, requires and it will be called the *maximum consecutive* system with respect to L_{jk}; S_n will stand for the sum of the weights of Σ_n.

We state and prove four lemmas below which will enable us to solve the problem of § 1. The reader should note that each of these lemmas expresses a property of Σ_n, though the first two of them suppose the *consecutive*, but not the *maximum consecutive*, property of Σ_n.

LEMMA I. *A consecutive system* σ_n, $n>1$, *contains*[1] *a* $1'$ *weight*, $w_1=1'$.

With $n>1$, suppose σ_n does not contain a $1'$ weight. Then $w_1\geqq 2'$ and w_2, which exists since $n>1$, must be at least as heavy as w_1. Therefore $w_2\geqq 2'$, and $s_n\geqq 4'$. We ask now how an object $W\equiv(ks_n-1)_0$, which is $\geqq 3_0$, can be weighed. We shall show that

(*a*) W must be weighted at -1;

(*b*) W cannot be weighed at -1 when $w_1\geqq 2'$, $n>1$.

To prove (*a*), let us suppose that W can be weighed at $-p$, where p is a positive integer $\geqq 2$. Placing W at $-p$, we create on the left arm of L_{jk} a moment of $pW=p(ks_n-1)$. Now the maximum moment that σ_n suffices to create on the right arm of L_{jk} is $ks_n=W+1$. But $pW>W+1$ since $W\geqq 3$ and $p\geqq 2$. Hence W cannot be weighed at $-p$; W must therefore be weighed at -1, which was to be proved.

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[1] At present the reader may convince himself that if $n=1$, the system σ_1 having $w_1=1'$ has not the maximum consecutive capacity for all pairs of values of j, k by observing that if $j\geqq 2$ and $k=1$, a $2'$ weight has a greater consecutive capacity than has a $1'$ weight. A complete treatment of the case $n=1$ will be given in §5.

To prove (b), we observe that W cannot be weighed at -1 if any one of the weights of σ_n is not at k; for, if any weight of σ_n is placed on L_{jk} at p, where p is an integer $<k$, the moment created on the right arm of L_{jk} will not exceed ks_n-2 since $w_1 \geqq 2'$. But if all weights of σ_n are placed at k, the moment ks_n ($>ks_n-1$) will be created on the right arm of L_{jk}. Hence (b) is true. The proof of Lemma I is now complete.

LEMMA II. *Let σ_n be a consecutive system; then every object of weight w which satisfies the relation $(ks_n/2)<w\leqq ks_n$ must be weighed at -1 by σ_n.*

For if an object of weight w were placed at -2, or at $-p$ where $p>2$, it would create on the left arm of L_{jk} a moment greater than ks_n, which is the maximum moment that can be formed on the right arm of L_{jk} with the weights of σ_n.

LEMMA III.

$$S_n \geqq \frac{(j+k+1)^n-1}{j+k}.$$

For Σ_n (Definition 6) must have at least as great a consecutive capacity with respect to L_{jk} as has the set (1) and therefore S_n must be $\geqq$ the sum (2) of the weights (1).

LEMMA IV. *Let $r=j+k+1$ and let m be an integer >1; then the following conditions on the weights of Σ_n are necessary (but not sufficient).*

$$w_{i+1} \leqq (r-1)(w_1+w_2+\cdots+w_i)+1 \text{ if } n>i+1,$$
$$(i=1,2,\cdots,\ m-1;\ m<n); \qquad (10_i)$$
$$w_{p+1} \leqq (r-1)(w_1+w_2+\cdots+w_p)+1 \text{ if } n>p,$$
$$(p=1,2,\cdots,m-1;\ m<n+1). \qquad (11_p)$$

We first prove (10_1).[1] Assume that (10_1) is not true. Then, since $w_1=1'$ (Lemma 1), $w_2>(r-1)+1$ if $n>2$. We shall prove that this assumption leads to a contradiction by showing that

(a) $W_1\equiv kS_n-(j+k)w_1-1$ must be weighed at -1;

(b) W_1 cannot be weighed at -1 if $w_2>(r-1)w_1+1=j+k+1$ and $n>2$.

To prove (a), we need only to show that with $n>2$

$$W_1 = kS_n-j-k-1 > kS_n/2;\ i.e.,\ kS_n/2 > j+k+1, \qquad (12)$$

so that W_1 must be weighed at -1 (Lemma 2). But since $n>2$, therefore $\geqq 3$, the last form of (12) follows directly from the two relations

$$S_n \geqq \frac{(j+k+1)^n-1}{j+k};\quad k\cdot\frac{(j+k+1)^n-1}{2(j+k)} > j+k+1;$$

[1] The notation (10_1) is used to indicate what formula (10_i) becomes when i is replaced by 1.

the former relation being true by Lemma III, and the latter holding for every pair of positive integers j, k. The relation (12) is therefore true; hence (a) is true for (10_1).

We observe now that if we place upon n in (10_1) only the condition $n>1$, which is the hypothesis on n in (11_1), the modified relation (10_1) is not true for all pairs of positive integers j, k and all choices of n. For example, it is not true for the case $L_{jk}=L_{21}$, $n=2$, where $w_1=1'$, $w_2=6'$. But if we use both hypotheses of (11_1), that $n>1$ and $k>1$, we can prove (a) for (11_1) as follows. Certainly

$$W_1 = kS_n - j - k - 1 > kS_n/2 \text{ if } kS_n/2 > j + k + 1, \ (n > 1, \ k > 1). \quad (13)$$

But the last inequality of (13) follows from the two relations

$$S_n \geqq \frac{(j+k+1)^2-1}{j+k}; \quad k\,\frac{(j+k+1)^2-1}{2(j+k)} > j+k+1;$$

the former relation being true by Lemma III, and the latter holding when, and only when, the hypothesis $k>1$ is satisfied. Therefore the inequalities (13) hold, and, by Lemma II, (a) is true for (11_1).

To prove (b) for (10_1), we observe that W_1 cannot be weighed at -1 without the use of w_2 and all weights of Σ_n which are $\geqq w_2$ at k, since $w_2>(r-1)w_1+1$ when $n>2$ by our hypothesis, which we are to prove false. But with all of these weights at k, the lightest weight that Σ_n suffices to weigh at -1 on L_{jk} is $kS_n-(j+k)w_1=kS_n-r+1$, which is heavier than W_1. We have thus arrived at a contradiction. This proves (b) for (10_1), and thus completes the proof of (10_1).

The proof of (b) for (11_1) is essentially the same as that for (10_1), the two proofs differing only in that $n>2$ in (10_1) while n is merely >1 in (11_1). Hence (b) for (11_1) is true. This completes the proof of (11_1).

Adding (10_1), (11_1) to our hypotheses on Σ_n, we can prove (10_2), (11_2), respectively; then, with (10_1), (10_2) and (11_1), (11_2) holding, we can prove (10_3) and (11_3), respectively; etc. We shall prove these statements by assuming that (10_1), (10_2), $\cdots$, (10_{m-2}) hold and that (11_1), (11_2), $\cdots$, (11_{m-2}) hold and then showing that on these assumptions (10_{m-1}) and (11_{m-1}), respectively, hold.

We first prove (10_{m-1}). Assume that (10_1), (10_2), $\cdots$, (10_{m-2}) are true, but that (10_{m-1}) is not true. Then, since $w_1=1'$ (Lemma I), $w_m>(r-1)(1+w_2+\cdots w_{m-1})+1$ when $n>m$. We shall prove that this assumption leads to a contradiction by showing that

(c) $W_{m-1}\equiv kS_n-(r-1)(w_1+w_2+\cdots+w_{m-1})-1$ must be weighed at -1;

(d) W_{m-1} cannot be weighed at -1 if $w_m>(r-1)(1+w_2+\cdots+w_{m-1})+1$ and $1<m<n$.

To prove (*c*), we need only to show that when $1<m<n$,

$$W_{m-1} > kS_n/2\ ;\quad i.e.,\quad kS/2_n > (r-1)(1+w_2+\cdots+w_{m-1})+1, \tag{14}$$

so that W_{m-1} must be weighed at -1 (Lemma II). But the last inequality of (14) follows directly from two relations: *first*, with $n \geqq m+1$,

$$S_n \geqq \frac{(j+k+1)^{m+1}-1}{j+k},$$

(Lemma III); *second*, as we shall presently prove,

$$k\cdot\frac{(j+k+1)^{m+1}-1}{2(j+k)} > (r-1)(1+w_2+\cdots+w_{m-1})+1. \tag{15}$$

To prove (15), we first make use of our assumption that (10_1), (10_2), $\cdots$, (10_{m-2}) hold and the fact that $w_1=1'$, and obtain the following relations:

$$\begin{aligned}
&w_1 = 1\ ;\\
&w_2 \quad \leqq (r-1)w_1+1, \text{ so that } w_2 \leqq r\ ;\\
&w_3 \quad \leqq (r-1)(w_1+w_2)+1 \leqq (r-1)(r+1)+1, \text{ so that } w_3 \leqq r^2\ ; \qquad (16)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&w_{m-1} \leqq (r-1)(1+r+r^2+\cdots+r^{m-3})+1, \text{ so that } w_{m-1} \leqq r^{m-2}.
\end{aligned}$$

Then from (16) we have

$$S_{m-1} \leqq \frac{r^{m-1}-1}{r-1} = \frac{(j+k+1)^{m-1}-1}{j+k}. \tag{17}$$

We now prove (15) by showing that it holds when every weight of the set $w_1, w_2, \cdots, w_{m-1}$ has its maximum value (see (16)), *i. e.*, when their sum (17) is $[(j+k+1)^{m-1}-1]/(j+k)$. Replacing $w_1+w_2+\cdots+w_{m-1}$ in (15) by this sum, we obtain

$$k\frac{(j+k+1)^{m+1}-1}{2(j+k)} > (j+k+1)^{m-1},$$

which may be written in the form $(j+k+1)^{m-1}[k(j+k)^2+2(k-1)(j+k)+k]>k$. Since this last inequality is true for every pair of positive integers j, k the relation (15) holds, and therefore (14) is true; hence (*c*) is true for (10_{m-1}).

We observe now that if we place upon n in (10_{m-1}) the condition $n>m-1$, which is the hypothesis on n in (11_{m-1}), the modified relation (10_{m-1}) is not true for all pairs of positive integers j, k and all choices of n. For example, it is not true for the case $n=m=3$, $L_{jk}=L_{31}$, where $w_1=1'$, $w_2=5'$, $w_3=39'$.

But if we use both hypotheses of (11_{m-1}), that $n>m-1$ *and* $k>1$, we can prove (c) for (11_{m-1}) as follows. Certainly

$$W_{m-1} > kS_n/2 \text{ if } kS_n/2 > (r-1)(1+w_2+\cdots+w_{m-1})+1. \tag{18}$$

But the last inequality of (18) follows from two inequalities: *first*, $S_n \geqq [(j+k+1)^m-1]/(j+k)$ when $n>m-1$ and $k>1$, which is true by Lemma III; *second*, as we presently prove

$$k\cdot\frac{(j+k+1)^m-1}{2(j+k)} > (r-1)(1+w_2+\cdots+w_{m-1})+1.$$

To prove the last inequality, we replace $1+w_2+\cdots+w_{m-1}$ in its right member by the maximum value of this sum (see (17)), and simplify the result. We thus obtain the inequality $(j+k+1)^{m-1}[k(j+k+1)-2(j+k)]>k$, which is true since $k>1$. Therefore (18) holds; hence (c) is true for (11_{m-1}).

To prove (d) for (10_{m-1}), we observe that $W_{m-1}=kS_n-(j+k)(1+w_2+\cdots+w_{m-1})-1$ cannot be weighed at -1 without the use of w_m and all weights of Σ_n which are $\geqq w_m$ at k, since $w_m>(j+k)(1+w_2+\cdots+w_{m-1})+1$ when $n>m$ by our hypothesis, which we are to prove false. But with all of these weights at k, the lightest object that Σ_n suffices to weigh at -1 on L_{jk} is $kS_n-(j+k)(1+w_2+\cdots+w_{m-1})$, which is heavier than W_{m-1}. We have thus arrived at a contradiction. This proves (d) for (10_{m-1}) and completes the proof of (10_{m-1}).

The proof of (d) for (11_{m-1}) differs from that just given for (10_{m-1}) only in that $n>m$ in (10_{m-1}) while n is merely greater than $m-1$ in (11_{m-1}). Therefore (d) is true for (11_{m-1}); hence (11_{m-1}) is true. This completes the proof that the conditions (10_1), (10_2), $\cdots$, (10_{m-1}) and (11_1), (11_2), $\cdots$, (11_{m-1}) are necessary. That they are not sufficient is obvious. Hence Lemma IV is true.

4. **Cases to be discussed in answering (B).** The statements of Lemma I and Lemma IV above imply that the case $n=1$ requires special consideration. In Lemma IV, the relations (10_1), $\cdots$, (10_{m-1}) imply also that $n=2$ is a special case; while relations (11_1), $\cdots$, (11_{m-1}) imply that $k=1$ is a special case. We shall answer (B) completely, without any overlapping in our argument, by treating the following cases: $n=1$, in § 5; $k=1$, $n=2$, in § 6; $k=1$, $n>2$, in § 7; $k>1$, $n>1$, in § 8.

5. **Discussion of (B) in the case $n=1$. Theorem I.** That $w_1=1'$ has a consecutive capacity with respect to every lever L_{jk} was shown just after Definition 5, § 1. But the example cited in the footnote to Lemma I, § 3, namely $w_1=2', j\geqq 2, k=1$, in which w_1 suffices to weigh 1_0 at -2 and 2_0 at -1 shows that a $1'$ weight does not have the maximum consecutive capacity for one weight with respect to every lever L_{jk}. This example leads us to seek

all pairs of special values of j, k for which a single weight $w_1 > 1'$ has a consecutive capacity. Having investigated the cases $w_1 = 1'$ and $w_1 = 2'$, we need only to consider pairs of positive integers j, k for which $w_1 > 2'$ has a consecutive capacity. Such a w_1 must suffice to weigh on L_{jk} all integral weights from 1_0 to $(kw_1)_0$ inclusive. In particular, then, w_1 must suffice to weigh $(kw_1 - 1)_0$. Since $w_1 \geqq 3'$, $kw_1 - 1 > kw_1/2$. Hence, by Lemma II, § 3, $(kw_1 - 1)_0$ must be weighed on L_{jk} at -1. Now the only weights which a consecutive system $\sigma_1 = w_1$ suffices to weigh at -1 are w_1, $2w_1$, $3w_1$, $\cdots$, $(k-1)w_1$, kw_1. But $kw_1 - 1$ does not appear in this set of weights, since the largest weight here which is less than kw_1 is $(k-1)w_1$, while $(k-1)w_1 < kw_1 - 1$ by the hypothesis that $k \geqq 1$ and $w_1 \geqq 3'$. Therefore $(kw_1 - 1)_0$ can not be weighed at -1. Hence there does not exist a $w_1 > 2'$ which has a consecutive capacity with respect to L_{jk}. We have therefore proved the following theorem.

THEOREM I: *The only case in which $w_1 > 1'$ has a consecutive capacity with respect to L_{jk} is when $k = 1, j \geqq 2$. Then $w_1 = 2'$ has not merely a consecutive capacity, but the maximum consecutive capacity for one weight on L_{jk}. In all other cases, $w_1 = 1'$ constitutes the only consecutive system σ_1 and therefore the maximum consecutive system Σ_1.*

6. **Discussion of (B) when $k=1$ and $n=2$. Theorem II.** By trial we find that if $j = k = 1$, the weights of the system Σ_2 are $w_1 = 1'$, $w_2 = 3'$; if $j = 2$, $k = 1$, the weights of Σ_2 are $w_1 = 1'$, $w_2 = 6'$. But when $j = 3, 4, 5, 6$, the respective systems Σ_2 are $(1', 9')$, $(1', 11')$, $(1', 13')$, $(1', 15')$. These facts lead us to expect that if $k = 1$, $n = 2$, and $j > 2$, then the weights of the system Σ_2 are $w_1 = 1'$ and $w_2 = (2j+3)'$. That $w_2 \leqq (2j+3)'$ follows from the fact that if $w_2 \geqq (2j+4)'$, the object $(w_2 - j - 1)_0$ cannot be weighed by w_1, w_2. Now if $w_2 = (2j+3)'$, can we weigh on L_{jk}, $j > 2$, objects of all weights from 1_0 to $(2j+4)_0$ inclusive? or since $w_1 = 1'$ and since the weights $w_2 - j$, $w_2 - j + 1$, $w_2 - j + 2$, $\cdots$, w_2, $w_2 + w_1 = w_2 + 1$ can be weighed at -1, can the weights from 2_0 to $(j+2)_0$ inclusive be weighed on L_{j1}? We answer this question as follows. The weights which $w_1 = 1$ and $w_2 = 2j+3$ suffice to weigh on L_{j1} at $-2, -3, \cdots, -j$ are included in the following $(j-1)$ sets of weights, each set being weighed[1] at the point indicated.

$$\frac{j+3}{i}, \quad \frac{j+4}{i}, \quad \frac{j+5}{i}, \cdots, \quad \frac{2j}{i}, \cdots, \quad \frac{2j+4}{i} \text{ at } -i, \qquad (i = 1, 2, \cdots, j). \quad (19i)$$

That these sets contain all integral weights from 2 in (19_j) to $j+2$ in (19_2) follows from two facts. *First*, (19_m), where m is an integer such that $2 \leqq m \leqq j-1$, con-

[1] Of course we are not interested in the application of the term "weighed" to the non-integral weights of the $(j-1)$ sets.

tains every integral weight u satisfying the relation $(j+3)/m \leqq u \leqq (2j+4)/m$. *Second*, since $(2j+4)/(m+1) > (j+3)/m$ under our hypothesis that $j \geqq 3$, with $2 \leqq m \leqq j-1$, there exists no integral weight which is heavier than $(2j+4)/(m+1)$ and lighter than $(j+3)/m$. This proves that we may choose the maximum value, $2j+3$, for w_2. We therefore have the following theorem.

THEOREM II. *When $k=1$ and $n=2$, the system Σ_2 is obtained by selecting the weights w_1, w_2, as follows: $w_1=1$, $w_2=3$ if $j=1$; $w_1=1$, $w_2=6$ if $j=2$; $w_1=1$, $w_2=2j+3$ if $j>2$.*

7. **Discussion of (B) when $k=1$ and $n>2$. Theorem III.** Putting $m=n-1$ in the relations (10_i), and determining w_1 by Lemma I, we obtain

$$w_1 = 1, \quad w_2 \leqq r, \quad w_3 \leqq r^2, \cdots, \quad w_{n-1} \leqq r^{n-2}. \tag{20}$$

In (20), we have upper bounds for $w_2, w_3, \cdots, w_{n-1}$. Suppose all of the equality signs of (20) held. Then the $(n-1)$ weights $w_1, w_2, \cdots, w_{n-1}$ would be

$$w_1 = 1, \quad w_2 = r, \quad w_3 = r^2, \cdots, \quad w_{n-1} = r^{n-2}; \tag{21}$$

they would have the sum $S_{n-1}=[(j+2)^{n-1}-1]/(j+1)$, as $k=1$; and they would, by § 2, suffice to weigh all objects from 1_0 to $(kS_{n-1})_0=(S_{n-1})_0$ inclusive. The next object to be weighed would then be $(S_{n-1}+1)_0$. Suppose this object and the $(n-1)$ weights (21) were placed at $-j$. They would create on the left arm of L_{j1} a moment of $M=j(2S_{n-1}+1)$. That M is an upper bound for w_n follows from two facts: (i) if w_n were taken $>M$, w_n could not be used with the weights (21) to weigh $(S_{n-1}+1)_0$, and hence w_n would form with the weights (21) a non-consecutive system; (ii) if in (20) not all of the equality signs held, the sum of the $(n-1)$ weights $w_1, w_2, \cdots, w_{n-1}$ would be a positive integral weight S where $S<S_{n-1}$. Then, as is easily seen by the type of argument which was used in (i), the weight w_n could not exceed $j(2S+1)$, and would therefore be $<M$.

We wish now to show that the system of weights

$$w_1 = 1, \quad w_2 = r, \quad w_3 = r^2, \cdots, \quad w_{n-1} = r^{n-2}, \quad w_n = j(2S_{n-1}+1), \qquad (r = j+2), \tag{22}$$

obtained by adding w_n to the system (21), has a consecutive capacity with respect to L_{j1}. The objects which can be weighed at -1 by the system (22) are those that can be weighed by the weights (21) alone and those which the system (22) suffices to weigh when w_n is placed at 1; the two sets of objects being all those

$$\begin{aligned} &\text{from } 1_0 \text{ to } (S_{n-1})_0 \text{ inclusive,} \\ &\text{and from } [j(S_{n-1}+1)]_0 \text{ to } [(2j+1)S_{n-1}+j]_0 \text{ inclusive,} \end{aligned} \tag{23}$$

respectively. The remaining objects which the system (22) suffices to weigh on L_{j1} are included in the following $(j-1)$ sets of weights, each set being weighed at the point indicated.

$$\frac{j(S_{n-1}+1)}{i}, \quad \frac{j(S_{n-1}+1)+1}{i}, \cdots, \quad \frac{2j(S_{n-1}+1)-2}{i}, \cdots,$$

$$\frac{(2j+1)S_{n-1}+j}{i} \text{ at } -i, \qquad (i=2,3,\cdots,j). \tag{24$_i$}$$

In order to show that the integral weights in $(24_2), \cdots, (24_j)$ together with those in (23) form a consecutive set running from 1 to $(2j+1)S_{n-1}+j$ inclusive, we must show that the sets $(24_2), \cdots, (24_j)$ contain all integral weights from $S_{n-1}+1$, which occurs in (24_j), to $j(S_{n-1}+1)-1$, which occurs in (24_2). That this is true can be proved by the method which was used in making a similar proof just after equations (19_i). Hence system (22) is a consecutive system of $n>2$ weights, and indeed the maximum consecutive one for the case $k=1$, $n>2$ since in (22) each w_p $(p=1, 2, \cdots, n)$ has attained its upper bound. We have therefore the following theorem.

THEOREM III. *When $k=1$ and $n>2$, the system Σ_n is obtained by selecting the weights* (22): $w_1=1$, $w_2=r, \cdots, w_{n-1}=r^{n-2}$, $w_n=j(2S_{n-1}+1)$, $(r=j+2)$, *where S_{n-1} is the sum of the first $(n-1)$ weights.*

8. **Discussion of (B) when $k>1$ and $n>1$. Theorem IV.** Putting $m=n$ in the relations (11_p), and determining w_1 for the case $n>1$ by Lemma I, § 3, we obtain

$$w_1=1, \quad w_2 \leqq r, \cdots, \quad w_{n-1} \leqq r^{n-2}, \quad w_n \leqq r^{n-1}, \quad (r=j+k+1, \ k>1). \tag{25}$$

Now from (25) and Lemma III, we have the respective inequalities

$$S_n \leqq \frac{(j+k+1)^n-1}{j+k}, \ k>1\,; \quad S_n \geqq \frac{(j+k+1)^n-1}{j+k}, \ k \geqq 1.$$

As $k>1$ in this discussion, these two inequalities must hold simultaneously; hence $S_n=[(j+k+1)^n-1]/(j+k)$, the value given for s_n in equation (2). This equality of course implies that the inequality signs in (25) must be discarded. Then the weights of Σ_n become identical with the set (1), and give an affirmative answer to (B). We have therefore proved the following theorem.

THEOREM IV. *When $k>1$ and $n>1$, Σ_n is obtained by selecting the weights* (1), *which suffice to weigh at -1 every positive integral weight which they suffice to weigh on L_{jk}.*

9. **Weights of system applicable only on the right arm of L_{jk}. Theorem V.** With this restriction, expressed above in condition (6), on the use of weights, let us seek the system Σ_n.

Reasoning here as we reasoned just before question (A), § 1, we find that the system σ_n defined by the weights

$$w_1 = 1,\ w_2 = k+1,\ \cdots,\ w_n = (k+1)^{n-1} \tag{26}$$

suffices to weigh on L_{jk} all objects from 1_0 to $[(k+1)^n-1]_0$ inclusive. The question here which corresponds to (B) of § 1 is: Does the system (26) have the maximum consecutive capacity with respect to L_{jk}? To answer this question, we proceed as we did in §§ 3–8 inclusive.

To make the discussion here which corresponds to that of § 3, we first observe that Lemma I and Lemma II of § 3 hold here. In place of Lemma III, we have

LEMMA IIIa. $$S_n \geqq \frac{(k+1)^n - 1}{k}.$$

In place of Lemma IV, we have

LEMMA IVa. *Let m be an integer >1; then the following conditions on the weights of Σ_n are necessary (but not sufficient).*

$$w_{i+1} \leqq k(w_1 + w_2 + \cdots + w_i) + 1 \text{ if } n > i+1,$$
$$(i = 1,2,\cdots,\ m-1;\ m < n);\quad (27_i)$$

$$w_{p+1} \leqq k(w_1 + w_2 + \cdots + w_p) + 1 \text{ if } n > p,$$
$$(p = 1,2,\cdots,\ m-1;\ m < n+1).\quad (28_p)$$

We omit the proofs of Lemma IIIa and Lemma IVa because they are easily made by the methods of § 3.

We shall solve our problem completely, without any overlapping in our argument, by treating separately the following cases. Case 1: $n=1$. Case 2: $k=1, j=1, n>1$. Case 3: $k=1, j>1, n>1$. Case 4: $k>1, n>1$.

Case 1. $n=1$. We obtain the result here which is given in Theorem I, since w_1 there could not be used except at the points $1, 2, \cdots, k$.

Case 2. $k=1, j=1, n>1$. By Lemma I, $w_1=1$. If $n=2$, we find by trial that $w_2=2$. If $n>2$, we put $m=n-1$ and $k=1$ in relation (27_i) and obtain

$$w_1 = 1,\ w_2 \leqq 2,\ w_3 \leqq 2^2,\ \cdots,\ w_{n-1} \leqq 2^{n-2}. \tag{29}$$

In (29) we have upper bounds for $w_2, \cdots, w_{n-1}$. It is necessary then to consider two possibilities: (*a*) when each of the $(n-2)$ weights attains its upper bound; (*b*) when one or more of the $(n-2)$ weights does not attain its upper bound.

(*a*) In this case the $(n-1)$ weights in (29) become $w_1=1, w_2=2, \cdots, w_{n-1}=2^{n-2}$, whose sum is $S_{n-1}=2^{n-1}-1$. Then Lemma IIIa implies that $w_n \geqq 2^{n-1}$. Suppose $w_n > 2^{n-1}$. Then $(w_n-1)_0$ cannot be weighed; for w_n-1 is an integer satisfying the inequality $S_{n-1} < w_n-1 < w_n$. This contradiction

shows that the inequality $w_n > 2^{n-1}$ does not hold. Therefore $w_n = 2^{n-1}$, and our system of weights is given by

$$w_1 = 1,\ w_2 = 2,\ \cdots,\ w_{n-1} = 2^{n-2},\ w_n = 2^{n-1}. \tag{30}$$

(*b*) Here the sum of the $(n-1)$ weights (29) is less than $2^{n-1}-1$. By Lemma IIIa, then, $w_n > 2^{n-1}$. But if this is true, $(w_n - 1)_0$ cannot be weighed. Hence no system σ_n satisfying (*b*) is the system Σ_n.

In case 2, therefore, Σ_n is given by (30).

Case 3. $k=1, j>1, n>1$. By considering possibilities (*a*) and (*b*) here, we can show by argument of the type just used that if $n>2$ the inequality $w_n > 2^{n-1}$ can hold in one, and only one, way, namely when $w_n = 2^{n-1}+1$. By trial we find that if $n=2$, the weights $w_1=1$, $w_2=3$ constitute Σ_2, so that this case is not an exception. We thus conclude that Σ_n is given by

$$w_1 = 1,\ w_2 = 2,\ w_3 = 2^2,\ \cdots,\ w_{n-1} = 2^{n-2},\ w_n = 2^{n-1} + 1. \tag{31}$$

Case 4. $k>1, n>1$. Putting $m=n$ in relation (28_p), and determining w_1 for this case by Lemma I, § 3, we obtain

$$w_1 = 1,\ w_2 \leqq k+1,\ w_3 \leqq (k+1)^2,\ \cdots,\ w_n \leqq ((k+1)^{n-1}. \tag{32}$$

Now from (32) and Lemma IIIa, we have the respective inequalities

$$S_n \leqq \frac{(k+1)^n - 1}{k},\ k > 1\ ;\ S_n \geqq \frac{(k+1)^n - 1}{k},\ k \geqq 1.$$

As $k>1$ in this discussion, these two inequalities must hold simultaneously; hence $S_n = [(k+1)^n - 1]/k$, the sum of the weights (26). This equality, of course, implies that every inequality sign in (32) must be discarded. We thus conclude that Σ_n for case 4 is given by (26).

Noting that the result in case 2 (see (30)) is a special case of the result for case 4, we have the following theorem.

THEOREM V. *Let the weights of the system Σ_n be applied only at the division points* 1, 2, $\cdots$, k *on the right arm of* L_{jk}. *Then, when* $n=1$, *Theorem* I, § 5, *applies here. When* $k=1, j=1$, *and* $n>1$, *or when* $k>1$, *and* $n>1$, *the weights of the system* Σ_n *are given by* (26). *When* $k=1, j>1$, *and* $n>1$, *the weights of* Σ_n *are given by* (31).

10. Repetitions. Weights of system applicable on either arm of L_{jk}. Theorem VI. *Definition* 7. Let $\Sigma_{m_r n}$ denote the maximum consecutive[1] system of positive integral weights of which there are n types $w_1, w_2, \cdots, w_n$ and m_r

[1] The term "consecutive" here, as in all of this paper, is applied only to a system of weights which suffices to weigh on L_{jk} all objects from 1_0 to k times the sum of the weights of the system, inclusive; the objects being placed at division points b_i which satisfy condition (5), §1.

weights of type w_r $(r=1, 2, \cdots, n)$, where $w_1 \leqq w_2 \leqq \cdots \leqq w_n$; let $S_{m_r n}$ denote the sum of the weights of $\Sigma_{m_r n}$. Further, let $\sigma_{m_r n}$ and $s_{m_r n}$ denote a consecutive (but not necessarily the maximum consecutive) system and its sum.

We begin our search for $\Sigma_{m_r n}$ as we first sought Σ_n in preceding sections, namely by finding a system of weights which the lever suggests as a possible consecutive system of large capacity.

To weigh 1_0, we select $w_1=1$. As there are m_1 weights w_1, they suffice to weigh on L_{jk} all objects from 1_0 to $(km_1)_0$ inclusive. The next object to be weighed is $(km_1+1)_0$. Placing it at -1 and the m_1 weights $w_1=1$ at $-j$, we create on the left arm of L_{jk} a moment of $(km_1+1)+jm_1=(j+k)m_1+1$. Hence we select $w_2=(j+k)m_1+1$. Continuing this reasoning by a method well illustrated above, we are led to believe that the system of weights which has m_r weights of the rth type below $(r=1, 2, \cdots, n)$,

$$w_1 = 1,\ w_2 = (j+k)m_1w_1 + 1, \cdots,$$
$$w_n = (j+k)(m_1w_1 + m_2w_2 + \cdots + m_{n-1}w_{n-1}) + 1, \quad (33)$$

is consecutive. Application of mathematical induction, as in § 2, shows that this is true, and indeed that the system $\sigma_{m_r n}$ whose n types are given in (33) suffices to weigh *at* -1 all objects from 1_0 to $[k(m_1w_1+m_2w_2+\cdots+m_nw_n)]_0$ inclusive.

In order to abbreviate a discussion which parallels that of §§ 3–8 above, we state here, without proof, the four lemmas which correspond to those established in § 3:

LEMMA Ib. *A consecutive system $\sigma_{m_r n}$ $(m_r>1$ for at least one value of $r)$ contains a weight $w_1=1$.*

LEMMA IIb. *Every object of weight w where $ks_{m_r n}/2<w\leqq ks_{m_r n}$ must be weighed by $\sigma_{m_r n}$ at -1.*

LEMMA IIIb. *$S_{m_r n}\geqq s_{m_r n}$, where $s_{m_r n}$ is the sum of the weights in the system whose n types are given in* (33).

LEMMA IVb. *Let s be an integer >1; then the following conditions on the weights of $\Sigma_{m_r n}$ are necessary (but not sufficient).*

$$w_1 = 1 \text{ (if } m_1n > 1);\ w_{i+1} \leqq (j+k)(m_1w_1 + \cdots + m_iw_i) + 1 \text{ if } n > i+1,$$
$$(i = 1, 2, \cdots,\ s-1;\ s<n);\quad (34_i)$$

$$w_1 = 1 \text{ (if } kn > 1);\ w_{p+1} \leqq (j+k)(m_1w_1 + \cdots + m_pw_p) + 1 \text{ if } n > p,$$
$$(p = 1, 2, \cdots,\ s-1;\ s<n+1). \quad (35_p)$$

We are supposing in this section that $m_1n>1$, as we have treated the case $m_1=n=1$ in §§ 1–8. If $n=1$, then $w_1=1$; otherwise the m_1 (>1) weights w_1

would not suffice to weigh $(km_1-1)_0$. It is easy to see now from (34_i) and (35_p) that our problem calls for consideration of the following cases: (*a*) when $n=2$ and $k=m_2=1$; (*b*) when $n>2$ and $k=m_n=1$; (*c*) when $km_n>1$. We consider these cases presently.

(*a*) *When* $n=2$ *and* $k=m_2=1$. We find by proceeding as in § 6 that $\Sigma_{m_r n}$ is obtained by selecting the following types of weights:

$$\text{with } w_1 = 1, \text{ then } w_2 = m_1(2j+1)+2 \text{ if } m_1 \leqq j-2;$$
$$\text{or } w_2 = j(2m_1+1) \text{ if } m_1 \geqq j-2. \qquad (36)$$

(*b*) *When* $n>2$ *and* $k=m_n=1$. By putting $s=n-1$ in relations (34_i), we obtain upper bounds for the first $(n-1)$ types of weights $w_1, w_2, \cdots, w_{n-1}$. Then proceeding as we did just before Theorem III, § 7, we find that by letting each of these types attain its upper bound, we can choose $w_n=j(2S_{m_r n-1}+1)$, where $S_{m_r n-1}$ is the sum of all weights of the first $(n-1)$ types. We find then that $\Sigma_{m_r n}$ has the following n types of weights

$$w_1 = 1, \; w_2 = (j+k)m_1 w_1 + 1, \cdots, \; w_{n-1} = (j+k)(m_1 w_1 + m_2 w_2 + \cdots + m_{n-2} w_{n-2}) + 1, \; w_n = j(2S_{m_r n-1}+1). \qquad (37)$$

(*c*) *When* $km_n>1$. Putting $s=n$ in relations (35_p) and proceeding as in § 8, we find that the system $\Sigma_{m_r n}$ for this case has the n types given in (33).

The results of this section are summarized in the following theorem.

THEOREM VI. *If* $n=1$, *the system* $\Sigma_{m_r 1}$ (*cf. Definition* 7, § 10) *consists of* m_1 (>1) *weights,* w_1, *each equal to* 1. *If* $n=2$ *and* $k=m_2=1$, $\Sigma_{m_r 2}$ *is obtained by selecting its two types of weights as described in* (36). *If* $n>2$ *and* $k=m_n=1$, $\Sigma_{m_r n}$ *has the* n *types of weights which are given by* (37). *If* $km_n>1$, *then* $\Sigma_{m_r n}$ *is obtained by selecting the* n *types of weights which are given by* (33); *in this case* $\Sigma_{m_r n}$ *suffices to weigh at* -1 *every positive integral weight which it suffices to weigh on* L_{jk}.

The following corollary to Theorem VI is easily deduced.

COROLLARY. *If* $m_1=m_2=\cdots=m_n=u$, $u>1$, *so that* m_r *has the same integral value* >1 *for each of the* n *values of* r, *the system* $\Sigma_{m_r n}\equiv\Sigma_{un}$ *is obtained by selecting* u *weights of each of the following* n *types* (*cf.* (33)):

$$w_1 = 1, \; w_2 = (j+k)u+1, \; w_3 = [(j+k)u+1]^2, \cdots, \; w_n = [(j+k)u+1]^{n-1}.$$

Furthermore, Σ_{un} *suffices to weigh at* -1 *every positive integral weight which it suffices to weigh on* L_{jk}.

11. **Repetitions. Weights of system applicable only on the right arm of L_{jk}.** Theorem VII. To obtain the system $\Sigma_{m_r n}$ in this case, we proceed as in § 10. We find that the lever suggests the n weights

$$w_1 = 1,\ w_2 = km_1w_1 + 1,\ w_3 = k(m_1w_1 + m_2w_2) + 1, \cdots,$$
$$w_n = k(m_1w_1 + m_2w_2 + \cdots + m_{n-1}w_{n-1}) + 1 \qquad (38)$$

as the types of a possible consecutive system $\sigma_{m_r n}$ of large capacity. It can be shown by mathematical induction that this system is consecutive, and indeed that it suffices to weigh at -1 all integral weights from 1_0 to k times the sum of its weights.

Lemma Ib and Lemma IIb of § 10 hold here. In place of Lemma IIIb and Lemma IVb of § 10, we have the following:

LEMMA IIIc. *$S_{m_r n} \geqq s_{m_r n}$, where $s_{m_r n}$ is the sum of the weights of the system $\sigma_{m_r n}$ whose n types of weights are given by* (38).

LEMMA IVc. *Let s be an integer >1; then the following conditions on the weights of the system $\Sigma_{m_r n}$ are necessary (but not sufficient).*

$$w_1 = 1 \text{ (if } m_1 n > 1);\ w_{i+1} \leqq k(m_1w_1 + \cdots + m_iw_i) + 1 \text{ if } n > i+1,$$
$$(i = 1,2,\cdots,\ s-1;\ s<n):$$
$$w_1 = 1 \text{ (if } kn > 1);\ w_{p+1} \leqq k(m_1w_1 + \cdots + m_pw_p) + 1 \text{ if } n > p,$$
$$(p = 1,2,\cdots,\ s-1;\ s < n+1).$$

Proceeding now as in § 10, we obtain the following theorem.

THEOREM VII. *Let the weights of the system $\Sigma_{m_r n}$ be applied only at the division points on the right arm of L_{jk}. Then when $n=1$, $\Sigma_{m_r n}$ consists of m_1 (>1) weights w_1 each equal to 1. If $n \geqq 2$ and $k = m_n = 1$, the first $(n-1)$ types of weights of $\Sigma_{m_r n}$ are given by the equations*

$$w_1 = 1,\quad w_2 = m_1 + 1,\ \cdots,\quad w_{n-1} = (m_1w_1 + m_2w_2 + \cdots + m_{n-2}w_{n-2}) + 1,$$

while $w_n = S_{m_r n-1} + 1$ if $j=1$ and $w_n = S_{m_r n-1} + 2$ if $j>1$, where $S_{m_r n-1}$ is the sum of all weights of the first $(n-1)$ types. When $km_n > 1$, the system $\Sigma_{m_r n}$ has the n types of weights which are given by (38); *in this case $\Sigma_{m_r n}$ suffices to weigh at -1 every positive integral weight which it suffices to weigh on L_{jk}.*

The following corollary to Theorem VII is easily obtained.

COROLLARY. *If $m_1 = m_2 = \cdots = m_n = v$, $v>1$, so that m_r has the same integral value for each of the n values of r, the system $\Sigma_{m_r n} \equiv \Sigma_{vn}$ is obtained by selecting v weights of each of the following n types (cf.* (38)):

$$w_1 = 1,\ w_2 = kv + 1,\ w_3 = (kv+1)^2,\ \cdots,\ w_n = (kv+1)^{n-1}.$$

Furthermore, Σ_{vn} suffices to weigh at -1 every positive integral weight which it suffices to weigh on L_{jk}.

12. **Problems solved. Problems proposed to the reader.** Theorems I, II, III, IV, and VI together constitute a solution of the following general Diophantine problem.

Let $w_1, w_2, \cdots, w_n$ be n types of positive integers. Select a positive number m_r of integers of type w_r $(r=1, 2, \cdots, n)$, and denote the sum of all the integers selected by $S_{m_r n}$. It is required to find the largest $S_{m_r n}$ for which there exists a consistent system of linear equations in $w_1, w_2, \cdots, w_n$ of the form

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \quad (i = 1,2, \cdots, kS_{m_r n}),$$

where the a's and b's are integers satisfying the conditions

$$-jm_r \leqq a_{ir} \leqq km_r \, (r = 1,2, \cdots, n), \quad -j \leqq b_i \leqq -1,$$

j and k being positive integers.

Theorems V and VII together constitute a solution of the problem which results when the condition on the a's above is replaced by

$$0 \leqq a_{ir} \leqq km_r, \qquad (r = 1,2, \cdots, n),$$

all other features of the above problem remaining the same.

At the suggestion of Professor D. R. Curtiss, we propose here three problems which we hope will interest the reader. The author attempted to solve Problem 1 before he made any progress on the present paper. The last two problems are due to Professor Curtiss.

Problem 1. Let $w_1, w_2, \cdots, w_n$ be n positive integers. It is required to find the largest positive integer p for which there exists a consistent system of linear equations of the form

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \quad (i = 1,2, \cdots, p),$$

where the a's and b's are integers satisfying the conditions

$$-j \leqq a_{iq} \leqq k, \quad (q = 1,2, \cdots, n), \quad -j \leqq b_i \leqq -1,$$

j and k being positive integers.

Problem 2. Given n objects of weights $a_1, a_2, \cdots, a_n$, which can be weighed (cf. Definition 3, § 1) on L_{jk} by use of n, but not less than n, positive integral weights $w_1, w_2, \cdots, w_n$: it is required to find the largest weight w that can occur in any set of n weights which suffice to weigh the n objects $a_1, a_2, \cdots, a_n$; in other words, to find the largest w that can occur in a solution of the system of linear equations in $w_1, w_2, \cdots, w_n$,

$$a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n + b_i i = 0, \qquad (i = 1,2, \cdots, n),$$

where the coefficients a_{iq} and b_i are as described in Problem 1 except that here for each i there must exist at least one q such that $a_{iq} \neq 0$.

Example. If $j=k=1$, if $n=3$, and if the three objects which cannot be weighed by two weights are a_1, a_2, a_3, then three w's which suffice to weigh a_1, a_2, a_3 are $w_1=a_2+a_3$, $w_2=a_1+a_2+a_3$, $w_3=a_1+a_2+2a_3$. If $a_1<a_2<a_3$, we assert that $w_3=a_1+a_2+2a_3$ is the largest w that can occur in a set of three w's which suffice to weigh a_1, a_2, a_3 on L_{11}.

Problem 3. Assuming as in Problem 2 that no $(n-1)$ of the w's suffice to weigh the n objects $a_1, a_2, \cdots, a_n$ on L_{jk}, find the number of sets of n weights w which will accomplish the result.

Example. If $j=k=1$ and if $a_1=8'$, $a_2=15'$, $a_3=53'$, we assert that there are more than 50, but less than 300, sets of three w's which suffice to weigh a_1, a_2, a_3 on L_{11}. Let the reader find how many such sets of w's exist. Further, with this choice of j, k, n, let him find the maximum number of solutions w_1, w_2, w_3 of Problem 3 for all positive integral w_1, w_2, w_3.

THE CLOCK PARADOX OF THE THEORY OF RELATIVITY

By LUISE LANGE, Chicago, Illinois

At superficial thought the very first consequence of the Lorentz transformation may appear paradoxical: that of two systems in uniform relative motion each one finds its own clocks to go faster than those of the other. Yet, of course, there is no paradox in this. As this situation is of basic importance for our further discussion we shall begin by briefly recalling it. Each of the two systems S and S' is thought of as fitted with a set of identical clocks placed at certain measured distances and synchronized in the following way: at the moment when stations $x=0$ and $x'=0$ pass each other a light signal is sent out from that point at times $t=t'=0$, and observers in S and S' at x and x' respectively are instructed to set their clocks on $t=x/c$ and $t'=x'/c$ on receiving the signal. Now an observer O' stationed at $x'=0$ passes $x=vt$ at time t. According to the Lorentz equations[1] his clock then shows

$$t' = (1 - v^2c^{-2})^{-1/2}(t - vc^{-2}x) = (1 - v^2c^{-2})^{-1/2}(t - v^2c^{-2}t) = t(1 - v^2c^{-2})^{1/2},$$

so that observers in S will declare it to run $(1-v^2c^{-2})^{1/2}$ times slower than theirs. O' however could answer that he does not agree with their view of simultaneity on which their conclusion concerning his time-lag rests; that according to his own standards clock C_1 at $x=vt$ did not show $t=0$ when he, O', passed $x=0$. For that place, which he with his velocity v reaches at

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[1] In case these should not be familiar, they may be found derived in every treatise on the theory of relativity.

$t' = (1-v^2c^{-2})^{1/2}t$, is located, relative to S', at $x' = v(1-v^2c^{-2})^{1/2}t = (1-v^2c^{-2})^{1/2}x$; and for $t'=0$, $x' = (1-v^2c^{-2})^{1/2}x$, the L. E. give

$$t = (1-v^2c^{-2})^{-1/2}(t' + vc^{-2}x') = (1-v^2c^{-2})^{-1/2}vc^{-2}(1-v^2c^{-2})^{1/2}x = vc^{-2}x = v^2c^{-2}t,$$

so that this clock C_1 during his journey from $x=0$ to $x=vt$ went on only by $t-v^2c^{-2}t = t(1-v^2c^{-2})$, that is $(1-v^2c^{-2})^{1/2}$ times *slower* than his own.

Thus it is the different view of simultaneity of distant events which causes opposite inferences to be drawn from the same clockreadings in the two systems. Yet in thus comparing the clocks in S and S' it is really only *one* clock of the one system which is always compared to a *set of synchronized clocks* in the other. A direct comparison of two individual clocks in uniform relative motion is indeed intrinsically impossible because, as they cannot meet more than once, they cannot be compared more than once by an observer in the neighborhood of both; hence all he can do is either —as explained before—combine with a second observer, in which case synchronism must be established between the two distant places by assuming that light travels with velocity c in either direction relative to them. Or if the first observer can continue to keep the moving clock in sight and thus compare it to his own he must yet subtract the time of light transmission, again on the same assumption. Hence not the march of clocks in two such systems is being compared, but the judgment of two observers concerning the time between two events (e. g. the departure of O' at $x=0$ and his arrival at $x=vt$), where this judgment involves more than the reading of the clocks to be compared.

Now one point is important: among the determinations of the time between two given events made from various systems there is one privileged, namely that measured by the observer present at both events (in our example O' who, in contrast to observers in S, is present at both his departure and arrival); for he uses for the measurement nothing but one clock, whereas all other observers obtain their result by combining the readings of two distant clocks defined to be synchronous; this time is also the only one which is *experienced* as time, a steady flow of one consciousness accompanying the movement of the clock-hands, whereas judged from S this time interval is merely a computed magnitude, one which by the nature of the case cannot be lived through. The supposed comparison described above was therefore one between "experienced" or "proper time" in the one system and "computed" in time in the other, not between "time" unqualified.

With these distinctions brought out we approach the starting point of the so-called paradox: It being felt that no comparison of the proper times of *two systems meeting only once* is possible, efforts were made to make the theory answer this question for two systems which do return to each other a second

time, such that two observers can directly compare the time between the two encounters.

In 1911 P. Langevin[1] formulated and solved the problem in the since well-known way: Peter stands on the earth, Paul, enclosed in a bullet, is fired away with tremendous velocity, but due to a suitable encounter with some star his path is directed back to the earth. On both journeys from and to the earth his clock is slow in comparison to those at rest to the earth; (as say the L. E.); hence in returning his time is behind earth-time in the ratio $(1-v^2c^{-2})^{-1/2} : 1$. If he stayed away two years according to his time that elapsed on the earth is, say, two hundred years, if the velocity with which he performed his cosmic journey was such that $(1-v^2c^{-2})^{1/2}=0.01$. While Paul has aged only slightly during his absence, the sixth generation of Peter's descendants are already populating the planet. (This result is frequently spoken of as Langevin's Paradox, but unjustly; for there is nothing self-contradictory in it, contrary as it may be to customary notions. It turned paradox only in the later development.)

As one notices, the accelerations in the movement of the traveler have been left out of account; it was treated as consisting of two uniform motions in opposite directions. Apparently to bring out this point with greater clearness, another French physicist[2] changed the formulation as follows: Paul leaves Peter at point A at time $t=t'=0$ with a velocity such that $(1-v^2c^{-2})^{1/2}=\frac{1}{2}$. He reaches point B at earth-time $t=2$, his own clock then reading $t'=1$ (inferred from the L. E.). Now he "suddenly jumps into another system animated with a uniform velocity $-v$ and thus travels back. Everything happens as on the first part of the trip, as the numerical value of v is unaltered; coming together again they state that the time which was "measured and experienced" as four hours by Peter was "measured and experienced" as only two hours by Paul; time for the observer traveling back and forth flowed half as fast as for the one standing still.

Yet an objection arose: why say that Peter stood still and that Paul moved, in a theory which asserts the equivalence of systems in relative motion, with the implication that every result must be reciprocal for the systems involved? Taking the bullet as system of reference it is the earth with Peter which moves back and forth; hence it is the earth-time that lags behind, and thus at their second encounter Paul finds Peter not in his grave, but practically unaltered since he lived through only 0.02 year.—But, most distressing of all, *both views are equally justified*, the time lived through by either one is both longer and shorter than that of the other. This is Langevin's paradox.

[1] Paper presented at a meeting at Bologne, 1911.

[2] In a letter to H. Bergson, quoted in *Durée et Simultanéité*, pp. 246 ff. its author is spoken of by Bergson as "one of the most distinguished physicists," but his name is not revealed.

Though much was made of this paradox, notably by philosophers,[1] relativist-physicists soon pointed out that it arises not as a consequence, but by one or several misapplications of the theory of relativity.

We consider it first in the form of the traveler who "changes trains," but to deal with motions free throughout from accelerations we avoid the jump from one system to the other; instead, we leave Paul attached to his first system and place another observer, James, on the one moving oppositely, who sets his clock in accordance with Paul's when he passes him, on $t''=t'=1$. Now we have to distinguish three different uniform systems, S, S', S''. The systems S' and S'' move relatively to S with velocities $\pm v$, and therefore S'' moves relatively to S' with velocity

$$u=\frac{-v-v}{1-\frac{v}{c^2}(-v)}=\frac{-2v}{1+\frac{v^2}{c^2}},$$

and, using again the numerical value $(1-v^2c^{-2})^{1/2}=\frac{1}{2}$, this gives $7u+8v=0$.

The conclusion now that time in the double-system $S'-S''$ flows twice as slow as in S was based on the consideration that, as both move with the same numerical velocity relatively to S, the time elapsing in either of them in going from A to B and back from B to A is the same. Yet it is only judged by the standards of simultaneity in S that the clocks in S' and S'' run synchronously; whereas judged from S' the proper time between two events in S'' is $(1-u^2c^{-2})^{-1/2}$ or 7 times slower than the time computed to have elapsed between them in S'-time, and *vice versa*. Therefore observer James judges: For Paul to travel from A to B took 1 hour S'-time, which is 7 hours S''-time; back from B to A in my system took 1 hour in S''-time, hence the two trips $\overline{AB}$ and $\overline{BA}$ together 8 hours; and Paul figures similarly in S'-time. Thus there is the usual reciprocal overestimation of the proper times in relatively moving systems, but no paradox, as no two of the three observers meet more than once.

Next, what difference does it make if one of the two systems undergoes an acceleration in reversing its velocity?[2] The answer is: it destroys the equivalence of the two systems such that they cannot any more be at will considered as moving or at rest. For so far we are dealing exclusively with the restricted theory of relativity which asserts the equivalence only of systems in uniform relative motion. Accelerations from its standpoint are absolute, they imply forces, they bring into effect phenomena of inertia. Paul, if he did not jump,

[1] Two interesting though strikingly different interpretations are given by J. Petzold, *Die Stellung der Relativitätstheorie in der geistigen Entwicklung der Menschheit*, 2nd. ed., Leipzig, 1923, and by H. Bergson, *Durée et Simultanéité, à propos de la théorie d'Einstein*, 2nd. ed., Paris, 1923.

[2] If it is the observer who "jumps" from one system to the other this involves accelerations of no less consequence than if the system itself reverses its velocity.

suffered a severe bump against the front-wall of his enclosure if the reversal was sudden, or a perpetual pull towards it if it lasted some time, whereas nothing of the kind happened to Peter, hence Paul *does* move and thus it is *his* time which flows slower, not, as the paradox asserts, Peter's as well.

Yet this argument is not quite satisfactory, for the two trips back and forth were uniform, during these periods the two systems therefore remained equivalent and they are the only ones considered in the computation. Hence one could proceed as follows: refer the two uniform motions to Paul at rest but in-between attribute to him a sudden change from the one system of reference to the other. The question is, does that bring back the paradox? Is it in this description again Peter's time which flows slower? It does not, as a short consideration shows. The first uniform motion of Peter's with velocity $-v$ lasts 1 hour according to Paul's clock, while only $\frac{1}{2}$ hour in Peter's time as judged by Paul, namely

$$t = (1 - v^2c^{-2})^{-1/2}(t' + vc^{-2}x') = (1 - v^2c^{-2})^{-1/2}(1 + vc^{-2}(-v)) = (1-v^2c^{-2})^{1/2} = \tfrac{1}{2}.$$

The same happens during the second uniform motion with velocity $+v$. But, in performing the change from the first to the second system of reference Paul's view of temporality, in particular of simultaneity of distant events changes. From the point of view of the first system the "jump" which according to his own time took place at 1:00, corresponded to 12:30 ($\frac{1}{2}$ hour after the zero point) on Peter's clock; from the point of view of the second system however it is otherwise. For this system (S'') the Lorentz transformation with regard to Peter's world (S) assumes the form

$$t - t_1 = (1 - v^2c^{-2})^{-1/2}\left[(t'' - t_1'') - vc^{-2}(x'' - x_1'')\right]$$

where $t_1 = 2$ (S-time at place of jump), $t_1'' = 1$, $x_1'' = 0$, $t'' = 1$, $x'' = -v$ (Peter's place referred to S'' at time of jump); for the assumed value $(1-v^2c^{-2})^{1/2} = \frac{1}{2}$ this gives $t = 3\frac{1}{2}$. Thus in this description Peter's clock runs during the first uniform motion from 12:00 to 12:30, then during the jump leaps ahead to 3:30, and goes on from 3:30 to 4:00 during the second uniform motion. The result therefore is the same as before: what was 4 hours for Peter was 2 hours for Paul, not *vice versa*.

This is satisfactory inasmuch as it denies the paradox and seems to reveal a physically objective slower flow of time in the non-inert system, that is one independent of the system of reference. Yet there appears to be a doubtful point left. The computations made from the two different viewpoints show a strange dissymmetry. The first ran: Peter at rest, Paul moving back and forth losing time. The second: Paul at rest, Peter moving away losing time; Paul changing system of reference, Peter *gaining* time; Paul at rest, Peter losing time. The period of acceleration, of decisive influence in the second

treatment, has been entirely disregarded in the first. It was condensed into an "instant" and then neglected. But why could not this "instant" of Peter's —for it is in his time that we are figuring—be anything but an instant in Paul's time, when it has just been shown that the instantaneous change of system of reference of Paul's filled, in his computation, $3\frac{1}{2}$ hours of Peter's time? Why not figure out? There it lies: the theory does not provide the means to deal with this point. Its only tools are the Lorentz equations, and these speak of transformations among uniformly moving systems. How can they be applied to essentially non-uniform motions? Of course one is tempted to use differential methods, to consider the accelerated motion as consisting of indefinitely many uniform ones, successive velocities differing only by an indefinitely small amount Δv. But that does not remove the difficulty; for this increase Δv has to be acquired in a time small of at least second order,—so to speak in between two small intervals of uniform motion Δt—which makes these accelerations both infinitely great and infinitely frequent. Therefore to define as is sometimes done,[1] the proper time of an accelerated system as

$$\int_{x_1 t_1}^{x_2 t_2} (1 - [v(t)]^2 c^{-2})^{1/2} dt$$

is a very doubtful procedure. The transition from a differential $dy = f(x)dx$ to the integral $y = \int f(x)dx$ presupposes that the summation of any two elements leads to an element of the same kind or quality; here however one adds the proper time of one inert system to the proper time of another inert system; to say that their sum is equal to the proper time of a non-inert system means to disregard the possible influence of the accelerations as much as Langevin did in his treatment.[2]

Hence, inasmuch as this point is not cleared up, we have to admit that the restricted theory of relativity does not give the answer to the question: What are, in comparison, the proper times of two systems meeting twice and is the result obtained independent of the system of reference?

[1] See e.g. A. Kopff, *Mathematical Theory of Relativity*, English ed. 1923, p. 51, von Laue, *Theory of Relativity*, French ed. 1924, p. 80, J. Becquerel, *Exposé Elémentaire de la Théorie d'Einstein*, Paris 1922, p. 56.

[2] It is interesting to note how various authors get around this point. Kopff (*op. cit.*) attaches to his proper time integral a little inconspicuous footnote saying that all effects due to acceleration have been disregarded, and in another connection that accelerations are assumed to have no influence on the march of a clock (p. 125). Why so, he does not say. M. Born, *Theory of Relativity*, (French ed. 1923, p. 249) suggests that the uniform part of the motion in Langevin's proposition may be made so long that the period of acceleration may be considered negligible, which would (*a*) restrict the result to such cases where this assumption is fulfilled, and (*b*) even there is doubtful as we shall point out later. Von Laue, (*op. cit.* p. 63) speaks of "quasi-stationary accelerations, that is such that the inner status of the clock depends always on its instantaneous velocity, not on its acceleration," which, in this connection, might be called a *petitio principii*, for the question is just whether there exist such convenient quasi-stationary accelerations.

What about the general theory? In it all systems in an arbitrary state of motion are equivalent. Paul bumping against his enclosure may indeed be considered at rest and Peter who does not experience any disturbance as in accelerated motion, if namely one assumes a suitable field of gravitation oppositely directed to the previous acceleration in which the formerly accelerating forces keep Paul at rest, while Peter "falls free." The observable phenomena under these circumstances are indeed the same as before: The relative acceleration, Paul's being drawn towards the one wall of his bullet (which from the first point of view we would have ascribed to his inertia, from the second to his weight), and Peter's undisturbedness. Can this general theory deal better with our problem? It is claimed that it can; e. g. Kopff (*op. cit.*) remarks (p. 125) that it "provides the explanation for the clock paradoxes which turn up in the restricted theory of relativity." Einstein gave the solution as follows:[1] Two clocks A and B move so as to meet twice. In the first description of the process A is considered at rest, B as acted upon by forces which, after it has moved away[2] from A with uniform velocity $+v$ accelerate it in the opposite direction until it has assumed a velocity $-v$ which with it travels back to A uniformly. The question as to the comparative proper times in the two systems is treated as by Langevin; in neglecting the period of acceleration a time-lag of B during the two uniform motions is derived, $t' = (1-v^2c^{-2})^{1/2}t$, which is approximately $t' = t - \frac{1}{2}v^2c^{-2}t$.

In the second description B is considered at rest; A first moves away uniformly with velocity $-v$. Then a homogeneous field of gravitation in direction $\overrightarrow{AB}$ is introduced in which the previous forces keep B at rest while A falls free in it until it has assumed the velocity $+v$. The field vanishes, and A moves uniformly back to B. During the uniform parts of the motion A lags behind B; if they lasted t' in B-time they are only $t = (1-v^2c^{-2})^{1/2}t'$ in A-time. But during the reversing of the velocity A is at a place of higher gravitational potential than B; according to the theory, that makes A run faster than B so long as the field lasts. If in B-time it lasts $\Delta t'$, then in A-time it lasts $\Delta t'(1+\Delta\phi/c^2)$, where $\Delta\phi$ is the difference in gravitational potential. Now $\Delta\phi = h\cdot\gamma$, where h is the distance of A and B (measured in system B), and γ is the gravitational acceleration. Also $h = v\cdot t'/2$, ($t'/2$ being the time of each uniform motion in B-time), and $\Delta t'\cdot\gamma = 2v$. Hence, substituting, $\Delta t'\cdot\Delta\phi/c^2 = v^2/c^2\cdot t'$. Thus A *gains* during the reversal of its velocity by

[1] Einstein, Dialog über die Einwände gegen die Relativitätstheorie, in *Die Naturwissenschaften*, vol. 16, 1918; treated at more detail in Kopff. *op. cit.*, Born, *op. cit.*

[2] In the above presentation we treat the two clocks as having at the beginning and end a uniform relative velocity v. In the treatment referred to they are at relative rest at the beginning and end, which necessitates two more periods of acceleration. As their influence, however, is not accounted for we cut them out in this way, which does not affect the problem but slightly simplifies the presentation.

$v^2/c^2 \cdot t'$. This overcompensates the previous time-loss. If one uses the approximation $(1-v^2c^{-2})^{1/2}=1-\frac{1}{2}v^2c^{-2}$ the total gain of A over B at its return is $\frac{1}{2}c^2c^{-2} \cdot t'$, which is the same as that obtained on p. 28 by considering A at rest. Thus the paradox is removed; the time-lag of B is shown to be physically objective, that is independent of the system of reference.

Much as this method of treating the problem appears to differ from that given on pp. 26 the underlying idea is essentially the same: though the motion is referred to that system which is subject to the influence of forces yet the reversing of velocity has to be accounted for in such a way as to leave unaltered the physical phenomena making their appearance during this period. Accounting for them either way, by saying that the observer changes his system of reference, or that a gravitational field appears causing the appearance of his weight, the effect is the same: during this period time in the other system leaps ahead so as to overcompensate the time-loss during the uniform parts of the motion.

Yet just because of this similarity we do not feel any better satisfied than before, for our previous difficulty has not been touched at all. Now as before the description with A (Peter) as system of reference obtains the time-lag for B (Paul) only under disregard of the period of acceleration. Einstein in his paper does not mention this point at all, his interest being seemingly centered on showing how the case can also be dealt with by means of a gravitational field; Kopff, reproducing the argument and apparently feeling the gap, limits himself to the remark "these forces may have no influence on the movement of the clock B" (*loc. cit.*). Why *may* they not? These forces represent the only physical differentiation of the two clocks; it is due to them only that the unsymmetrical result as regards the two clocks is acceptable. Projected onto the other end by means of a fictitious gravitational field they swing the time-balance in favor of A, and yet, we are asked to *assume* that they have no influence on the clock on which they act. Moreover it appears that also Born's motivation for this omission (see footnote p. 27) is untenable; for the longer the *uniform* part of the motion the greater is the time-gain of A during the acceleration in the gravitational field. Why is not the same possible for B during its neglected acceleration? And again, why assume instead of compute? Apparently because also the general theory does not provide the means to deal with the influence of an acceleration not in a gravitational field. Nor will it do to say that only one description (with B as system of reference) is necessary to ascertain what happens, both versions referring to the same physical process; for, from the standpoint of the paradox, this simply means to beg the question.

Thus our survey has led us to the conclusion that the question regarding the difference in duration experienced in two relatively moving systems

between two encounters is as yet unanswered. It may be, as many relativists assert, that in worlds of different state of motion the *derôlement* of time is different, like on motion pictures reeled through slower and faster, with the inert systems being the fastest ones, but we do not see that the complete proof for it has as yet been given.

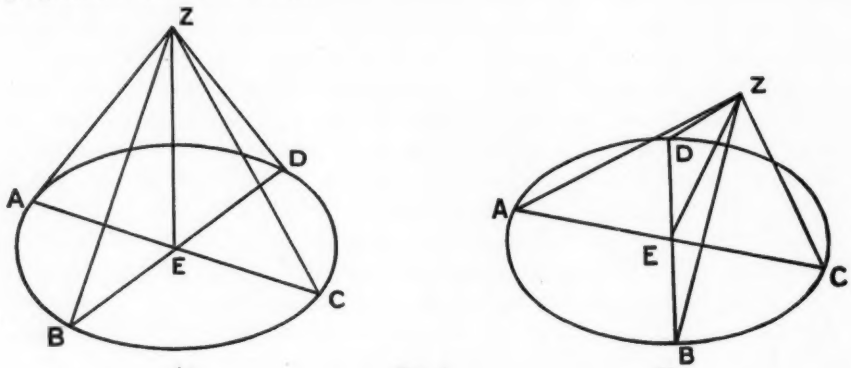
THE CIRCLE IN EUCLID'S TREATMENT OF OPTICS

By C. T. RUDDICK, University of Pennsylvania

The study of physical optics yields many problems of interest to the mathematical investigator. This is especially true of the first known work on the subject, that of Euclid, which has been preserved in the "Collections" of Pappus. Here are brought to light some properties of the circle as it *appears* to the eye in different positions, particularly in perspective.

It is the purpose of this paper to discuss some of the more general results obtained, and to reproduce in condensed form the treatment of the apparent ellipse formed by a circle in perspective. The work is essentially that of Euclid and Pappus, but arranged and greatly condensed to allow briefness and clarity of presentation. The propositions themselves are literal translations by Dr. J. H. Weaver, then of the University of Pennsylvania, now of Ohio State, from the parallel Greek and Latin, edition Hultsch of Pappus's "Collections."

The basis of the work is the principle that the apparent length of a line depends upon the angle which it subtends at the eye. From this the first proposition follows at once.



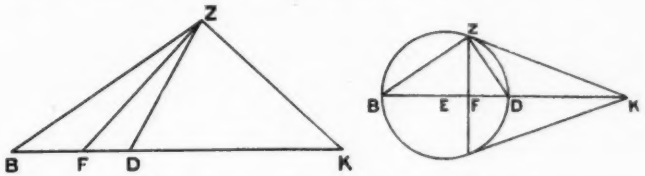
(*a*) FIG. 1 (*b*)

PROP. A. "Let there be a circle whose center is E, and from E let EZ be drawn perpendicular to the plane of the circle, then if the eye is placed in the line EZ, the diameters of the circle appear equal." (Fig. 1a)

This is of course obvious, since at any point on EZ the diameters will subtend equal angles, from a simple theorem of geometry.

"But let EZ be non-perpendicular, and let it be equal to a semidiameter, then if the eye is placed at Z, the diameters will appear equal." (Fig. 1b). This is equally manifest, since for any diameter BED, BE, ED, EZ are equal, hence the angle at Z is always right. Since the locus of the point Z is of course the surface of a hemisphere, it appears that all the diameters of the base of a hemisphere will seem equal to the eye placed at any point on the surface.

PROP. B. "If there is a circle, and from the center of it a straight line is erected, which is neither perpendicular to the plane nor equal to half a diameter of the circle, and at the end of this the eye be placed, the diameters of the circle will appear unequal." Two particular diameters appear as maximum and minimum, and the others are equal only in pairs. The proofs are based on an extensive array of lemmas, to the end that the angles subtended at the eye are proved unequal. The circle, viewed from such a position, appears of course to be an ellipse, and some of the peculiar properties of this ellipse will now be discussed. Much of the work to follow depends on a single lemma, which appears in Pappus thus: "Let $BK : KD = BF : FD$, and $\angle BZF = \angle FZD$, and KZ be drawn, then the angle FZK is right." Fig. 2a.



(*a*) FIG. 2 (*b*)

The theorem really amounts to the statement that the internal and external bisectors of the vertex angle of a triangle, along with the adjacent sides, cut the base in a range of points whose cross-ratio is harmonic. It should be noticed that tangents to a circle at the extremities of a chord perpendicular to the diameter effect the same division on the diameter. Fig. 2b.

Now (Fig. 3), consider the circle ABC, with the eye placed at Z. Drop ZH perpendicular to the plane of the circle, and draw $HDEB$ through the center E, intersecting the circumference in B, D.
Draw BZ, ZD, and bisect $\angle BZD$ with ZF, meeting the plane in F. Draw $AFC \perp BD$ at F, and tangents KA, KC, meeting BD in K.

Then these things are true: 1. ABC appears to be an ellipse, of which the center is at F, and the conjugate axes are AC and BD. Ordinates drawn to BD will be, and *appear* to be, parallel to AC. 2. Ordinates to AC are drawn from K, but *appear* parallel to BD.

1. Draw AZ, ZC, then $\angle AZF = \angle FZC$ and AF appears to equal FC. Also $\angle BZD = \angle FZD$ by construction, hence BF appears to equal FD.

Now any line LFM appears halved at F. Draw $LNXK$, KM, MPX, MZ, XZ, NZ, LZ, KZ. Because of the tangents KA and KC, as above, $BK : KD = BF : FD$ and $\angle BZF = \angle FZD$ by construction so $\angle FZK$ is right. Since plane $FZK \perp AZC$ then $ZK \perp$ plane AZC, and $\angle NZK$ is right. And since $\angle LFC = \angle CFX$, being measured by equal arcs, and $\angle CFK$ is right, then $LK : KX = LN : NX$.

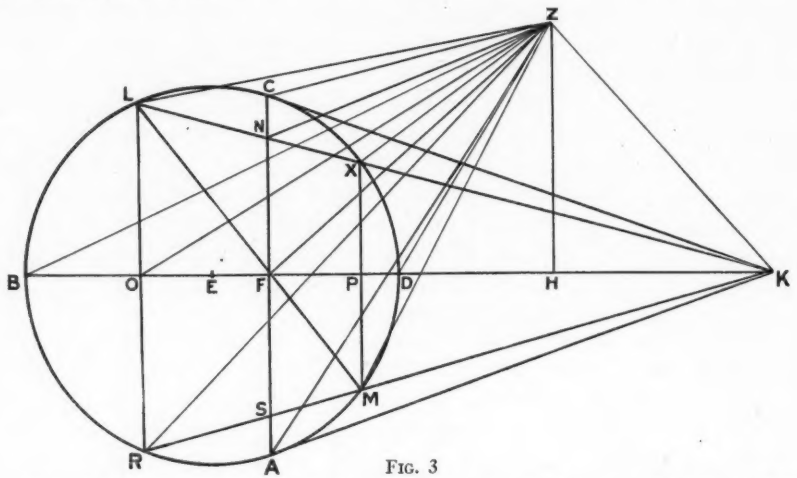


FIG. 3

Therefore, as a converse of the theorem stated above, $\angle LZN = \angle NZX$, and LN appears to equal NX. Also, because $\angle LZN = \angle NZX$ then $LZ : ZX = LN : NX$ and $ZM = ZX$, since $XM \parallel AC$, so $LZ : ZM = LN : NX$. But $LN : NX = LF : FM$, since $NF \parallel MX$, therefore $LZ : ZM = LF : FM$ and $\angle LZF = \angle FZM$. Thence LF appears to equal FM, and any line through F appears to be halved by it.

2. Now lines through the circle from K appear to be parallel. Draw $LO \perp BK$ and produce to R. Draw FZ, ZP, ZR. Since $LK : KX = LN : NX$ and $LN : NX = LZ : ZX$ and $LK : KX = LR : XM$, as shown above, then $LR : XM = LZ : ZX$. And, since $LZ = ZR$ and $ZX = ZM$, this becomes $LR : XM = ZR : ZM$.

Hence the isosceles triangles LZR and XMZ are similar, since their sides are all proportional, so that $\angle LZR$ equals $\angle XZM$. From this it is evident that $\angle LZO$ equals $\angle XZP$, therefore OL appears to equal PX. So LX appears parallel to BD, since perpendiculars between them appear equal.[1]

[1] It seems probable, from the method of propositions fifty-two, fifty-three, and fifty-four of this section of the sixth book of the "Collections," that this latter part of the discussion may be the work of Pappus himself, and not a part of the "Optics" of Euclid.

The question at once arises as to how two lines can appear parallel when we can see them meet at the point K. The investigation may be carried further by asking if all perpendiculars between LX and BD, or, more generally, between LK and BK, will appear equal. Taking the line FN for example, a new circle may be drawn through the four points N, X, M, S. Looking at this circle with the eye still at Z, a new point, say F', will appear to be the center, and since $\angle FZK$ is right, a new point K' will result, and lines from K' through this circle should appear parallel. But the line NXK does not pass through K', so NF will not appear to equal XP. The same reasoning applies to any other perpendicular.

So the appearance of such a line is uncertain. Experiment seems to show that it does appear parallel. Common sense suggests that it would appear curved instead of straight. The answer may lie in the construction of the eye. If it is capable of viewing the ensemble as a whole, the line perhaps should appear curved. But if the eye functions discontinuously, rather than continuously, it would seem that what has been demonstrated is true. That is, looking first at one and then at the other of the perpendiculars, they would appear equal, and the visual image retained would be one of parallel lines. Thus the phenomenon would reduce to an optical illusion.

ON PRODUCTS WHOSE DIGITS ARE CYCLICAL PERMUTATIONS OF THE DIGITS OF THE MULTIPLICAND

By ROBERT E. MORITZ, University of Washington

1. Introduction. It has been known for some years that the number 142857 has a property shared by no other number expressed in the common scale of notation, namely this, that it is the only number for which a multiplier of the same number of digits can be found such that the partial products consist of the numbers formed by cyclically permuting the digits of the number 142857.

$$142857 \times 1 = 142857$$
$$142857 \times 3 = 428571$$
$$142857 \times 2 = 285714$$
$$142857 \times 6 = 857142$$
$$142857 \times 4 = 571428$$
$$142857 \times 5 = 714285$$

The following paper deals with a general theorem from which the property in question follows as a very special case. The theorem, based on the most elementary algebraic relations, reveals many surprising results as, for instance, that the sixteen numbers 0588, 2352, 9411, $7647 \times k$, where k has the values

1, 2, 3, $\cdots$, 16, are formed by the cyclical permutations of the digits of the multiplicand.

2. Let s, p, and q be any three integers satisfying the relation

$$s^q \equiv 1 \pmod{p}. \tag{1}$$

Furthermore, let $d_1, d_2, \cdots, d_q$ represent the successive digits in the scale of notation, radix s, and $r_1, r_2, \cdots, r_q$ the corresponding remainders resulting from the division of the successive powers of s by p, thus

$$\begin{aligned}
s &= d_1 p + r_1, \\
s^2 &= (d_1 p + r_1)s = (d_1 s + d_2)p + r_2, \\
s^3 &= (d_1 s^2 + d_2 s + d_3)p + r_3, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
s^k &= (d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k)p + r_k, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
s^q &= (d_1 s^{q-1} + d_2 s^{q-2} + \cdots + d_{q-1}s + d_q)p + 1
\end{aligned} \tag{2}$$

Note: $r_q = 1$, because of the assumed relation (1).

From (2) we have

$$d_1 s^{q-1} + d_2 s^{q-2} + \cdots + d_{q-1}s + d_q = (s^q - 1)/p \tag{3}$$

and

$$r_k = s^k - (d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k)p. \tag{4}$$

Let N be the number represented by either side of equation (3). Then, from (3) and (4), the number $N \cdot r_k$ is equal to

$$(d_1 s^{q-1} + d_2 s^{q-2} + \cdots + d_{q-1}s + d_q)s^k - (s^q - 1)(d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k)$$

which reduces to

$$d_{k+1}s^{q-1} + d_{k+2}s^{q-2} + \cdots + d_q s^k + d_1 s^{k-1} + d_2 s^{k-2} + \cdots + d_{k-1}s + d_k$$

a number whose digits, in the s-scale of notation, are the k-th cyclical permutation of the digits of (3).

The foregoing result may be expressed in the form of a theorem as follows:

Theorem. Let $d_1, d_2, \cdots, d_q$ denote the successive digits, in the s-scale of notation, of a number N, obtained by dividing $s^q - 1$ by one of its divisors.[1] *Let r_k, $k < q$, be the remainder obtained on dividing s^k by this same divisor. Then the product $N \cdot r_k$, when expressed in the s-scale of notation, will have for its successive digits the k-th cyclical permutation of the digits of N.*

[1] When the quotient has less than q digits the deficiency must be supplied with zeros. Thus $(10^3-1)/111 = 9 = 0 \cdot 10^2 + 0 \cdot 10 + 9$; that is $d_1 = 0$, $d_2 = 0$, $d_3 = 9$.

3. Examples. In the denary scale of notation, $10^6-1=142857\times 7$, and 10^3 divided by 7 leaves a remainder 6; hence we have $142857\times 6=857142$ whose digits are the third cyclical permutation of the digits 142857.

Again, $10^6-1=076923\times 13$, and 10^5 divided by 13 leaves the remainder 4; hence $076923\times 4=307692$ whose digits are the fifth cyclical permutation of the digits 076923.

In the duodenary scale of notation, using t and e to denote ten and eleven,

$$(10^{14}-1)/15 = 0\ 857\ 921\ 4e3\ 642\ 9t7$$

$$r_1 = 10,\ 0\ 857\ 921\ 4e3\ 642\ 9t7 \times 10 = 8\ 579\ 214\ e36\ 429\ t70$$

$$r_2 = 8,\ 0\ 857\ 921\ 4e3\ 642\ 9t7 \times 8 = 5\ 792\ 14e\ 364\ 29t\ 708$$

$$r_3 = e,\ 0\ 857\ 921\ 4e3\ 642\ 9t7 \times e = 7\ 921\ 4e3\ 642\ 9t7\ 085$$

. . . . . .

$$r_{13} = t,\ 0\ 857\ 921\ 4e3\ 642\ 9t7 \times t = 7\ 085\ 792\ 14e\ 364\ 29t$$

$$r_{14} = 1.$$

It is obvious that the problem of finding the total number of products possessing the property set forth in our theorem resolves itself to the problem of the complete factorization of s^q-1 for given values of s and q. In the denary scale of notation, when $q=5$, $s^q-1=99999$, the distinct factors of which, not counting unity or the number itself, are 3, 9, 41, 123, 271, 369, 813, 2439, 11111, 33333. To each factor correspond theoretically $q-1=4$ products having the property in question. But whenever r_k is some power of 10 the result is trivial. The remaining significant results are

$$N = (10^5-1)/41 = 02439,\ r_2 = 18,\ r_3 = 16,\ r_4 = 37;$$

$$N = (10^5-1)/123 = 00813,\ r_3 = 16,\ r_4 = 37;$$

$$N = (10^5-1)/271 = 00369,\ r_3 = 187,\ r_4 = 244;$$

$$N = (10^5-1)/369 = 00271,\ r_3 = 262,\ r_4 = 37;$$

$$N = (10^5-1)/813 = 00123,\ r_3 = 187,\ r_4 = 244;$$

$$N = (10^5-1)/2439 = 00041,\ r_4 = 244.$$

Of special interest is the case in which p is an odd prime and s a primitive root of p. Then $s^{p-1}-1\equiv 0$, mod. p, and the successive powers of s when divided by p leave as remainders some permutation of the numbers $1, 2, \cdots, p-1$. If then we write $(s^{p-1}-1)/p=Q$, our theorem shows that the multiples $1Q, 2Q, \cdots, (p-1)Q$, when expressed in the s-scale of notation, have for digits the $p-1$ cyclical permutations of the digits of Q, the digits of nQ being the k-th cyclical permutation of the digits of Q, where k is the index belonging to n.[1]

[1] This result was arrived at by quite a different method by Bachmann, Zeitschrift für Mathematik und Physik, vol. 36 (1891), pp. 381–383.

In the denary scale of notation the highest digit is 9; p must therefore be less than 10. Now the only prime number less than 10, of which 10 is a primitive root, is 7; hence our corollary yields only one number which, when expressed in the denary scale of notation, possesses the afore-mentioned property. This number is $(10^6-1)/7=142857$.

When 12 is the radix of the scale of notation, there are two numbers Q such that every cyclical permutation of the digits of Q gives rise to a multiple of Q, for 12 is a primitive root of both 5 and 7. The two numbers are therefore $(12^4-1)/5=4147=2497$ in the duodenary scale, and $(12^6-1)/7=426569=186t35$ in the duodenary scale. The partial products of 2497 by 2431 and of $186t35$ by 546231, the radix being 12, are respectively as indicated below.

2497	$186t35$
2431	546231
4972	$86t351$
9724	$6t3518$
7249	$t35186$
2479	$35186t$
	$5186t3$
	$186t35$

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

A PERFECT NON-DENSE POINT SET

By RAYMOND GARVER, University of Rochester.

1. Introduction. If G is a linear point set, and P is a point such that every neighborhood of it contains points of G other than P itself, P is called a limit point (or an accumulation point) of the set G. If G contains all its limit points, it is said to be a closed set. If every point of G is a limit point of the set, G is said to be dense in itself. A set with both of these properties is a perfect set.

Now if a part of G is contained in a closed interval (a, b), and if every subinterval of (a, b) contains points of G, we say that G is everywhere dense in (a, b). If G has this property it is also dense in itself, provided that no points of G lie outside (a, b).[1] However, the converse is not true, as can be shown by examples. In fact, we can go further and define a set which is both perfect and

[1] See Cantor, Mathematische Annalen, vol. 23, p. 473.

non-dense; that is, *no* interval exists in which G is everywhere dense. The first such example was constructed by Cantor,[1] and others have been given. However the following example, which is similar to Cantor's, seems to illustrate the various properties of the set a little more plainly than do the others.

2. A Perfect Non-dense Set. The set G is defined to be the set of points (or real numbers) given by $. a_1a_2 \cdots a_n \cdots$, where each a_i is either 0 or 9. The set is thus contained in the interval (0, 1) and can easily be shown to be a perfect, non-dense set.

(a) *The set is dense in itself.* That is, any $P = . a_1a_2 \cdots a_n \cdots$ is a limit point of the set. For if we consider a neighborhood of P which extends at least a distance of $1/10^n$ to each side of P, we see that the point $Q = . a_1a_2 \cdots a_nb_{n+1}b_{n+2} \cdots$, where each b_i is either 0 or 9, but at least one b_i is different from the corresponding a_i in P, lies in this neighborhood, belongs to the set G, and is distinct from P.

(b) *The set is closed.* For consider a point $R = . a_1a_2 \cdots a_nc_{n+1} \cdots$, where R does not belong to G, and c_{n+1} is the first digit not a zero or nine. (If $c_{n+1}=8$, not all the succeeding digits are to be nines, or if $c_{n+1}=1$, not all the succeeding digits are to be zeros; in either of these cases R would belong to the set.) By definition, G contains no points between $(. a_1a_2 \cdots a_n0999 \cdots = . a_1a_2 \cdots a_n1)$ and $. a_1a_2 \cdots a_n9$. Hence we can take a neighborhood of R so small that it will certainly contain no point of G; that is, R can not be a limit point.

(c) *The set is non-dense in* (0, 1). We consider the sub-intervals of (0, 1) which contain no points of G in their interior. The largest is the interval $(.09999 \cdots, .90000 \cdots)$ or (.1, .9), of length .8. Next are two of equal size, $(.00999 \cdots, .09000 \cdots) = (.01, .09)$ and $(.90999 \cdots, .9900 \cdots) = (.91, .99)$. In general, no points of G lie between $(. a_1a_2 \cdots a_n0999 \cdots, = . a_1a_2 \cdots a_n1)$ and $. a_1a_2 \cdots a_n9$. There are 2^n such intervals, since each of $a_1, \cdots a_n$ may be either 0 or 9, and each of the intervals is of length $8/10^{n+1}$. The infinite sum of all these intervals is that of a geometric series whose first term is .8 and common ratio .2; it is therefore 1, the length of the interval. This set of non-overlapping intervals is therefore everywhere dense in (0, 1), and the set of points G must certainly be non-dense.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Statistical Methods, Applied to Economics and Business. By F. C. MILLS. New York, Henry Holt and Company, 1924. xvi+604 pages.

This book is essentially for the student of economics and business and the fundamental facts are written up in considerable detail, presumably on the

[1] See Cantor, Mathematische Annalen, vol. 21, p. 590.

hypothesis that much of the matter, even though elementary, is new to the reader. The first sixty pages develop the classes of business activity and the idea of the quantitative character of many economic and business problems. Attention is then called to statistical methods and problems of internal and external administration. Fifty pages are devoted to graphic presentation, including the standard rules proposed by the Joint Committee on Standards for Graphic Presentation. The next eighty-five pages are devoted to frequency distributions and all the usual ideas concerning such distributions together with the various types of averages are described. Only twenty pages are given to the development of dispersion, standard deviation, skewness and probable error and other measures of variation. It would hardly seem that the student would get much of an idea of the important constants characterizing frequency distributions from this account. Index numbers of prices occupy over eighty pages and the author goes into much detail concerning index numbers and their application to prices. In connection with the discussions concerning the various formulas for calculating index numbers and the selection of a so-called "ideal" index, one cannot help but feel that the choice in this and other texts is based upon the personal preferences and notions of the authors rather than upon exact methods of reaching the best results. Perhaps this must be so in the very nature of the concept. In a later chapter index numbers of physical volumes are also developed and here again each writer has his own method of defining the index. It is quite interesting, but not very convincing, to read over the various papers by economists with the general title "A new index, etc." They have invented a greater variety of index number formulas than Heinz has of pickles. Almost one hundred pages are given to the analysis of time series through the measurement of trend, and seasonal and cyclical fluctuations. The usual ideas of moving averages (which by the way is nothing more or less than the smoothing of rough series by various graduation formulas) are developed in much detail and the resulting graduated curves are then subjected to fitting by some mathematical formula, usually one for a linear, quadratic, cubic or logarithmic curve.

Over one hundred and fifty pages are devoted to the measurement of relationship and for this purpose the theory of linear correlation is developed and applied to various types of economic statistics. Applications of correlations are made in the next chapter to the measurements of relationship between time series and in the following chapter the elements of non-linear correlations are developed including an account of the correlation ratio. Having developed the necessary concepts in the preceding chapters the author then makes an application to the problem of estimation together with the standard error of the estimate. The index of correlation based on logarithmic values of the variables also receives attention in this chapter. A short chapter on multiple and partial correlation is also included.

The chapters which have been put at the end dealing with elementary probabilities and the normal curve of error, statistical induction and the problem of sampling seem to the reviewer to be out of place. If they had appeared towards the beginning of the book the student would undoubtedly have a much clearer knowledge of what is behind the various formulas which he has been using throughout the book so freely. The fundamental conception of statistics as embodied in the relations between the theory of probability and random sampling should be developed early in any course in mathematical statistics before important applications are taken up. No student of any applied subject can make intelligent applications of his formulas until he has first grasped these fundamentals. It is a curious fact that most writers in non-mathematical departments of investigation plainly avoid this basic mode of treatment or development of the subject, most likely in order to dodge the mathematics. It is unfortunate that this situation persists in so many texts, because it cannot help but turn out a crop of statisticians who have acquired a lot of confidence in the manipulation of formulas which they do not understand, a confidence which apparently entitles them to deal with the applications in various lines of investigation—including economics and business. Appendix A is a good illustration of this point. It is entitled "The Method of Least Squares as Applied to certain Statistical Problems." Of course, if the student has not digested this Appendix, he cannot possibly have a clear understanding of many of the applications which the author has made in his book. This is particularly true in the matter of correlation and the measurement of relationships. The author has developed this subject fairly well and it is unfortunate that it is not a part of the body of the book and required of all students rather than something relegated to the end to be omitted if found too difficult—in other words if there is too much mathematics.

Appendix B is a glossary of symbols used in formulas in the development of mathematical statistics as applied to frequency distribution constants, correlations constants, curve fitting constants, and so on. It seems to the reviewer that all these symbols should have been defined and developed more fully in the body of the book instead of being shoved into a glossary or appendix at the end. A very good list of references appears at the end of the volume and also at the end of each chapter. In view of the fact that so much of the mathematics is left unexplained, it might be helpful to add to this list some of the more important elementary texts on algebra, calculus, mechanics and the theory of probability. This at least would show the reader where he could find some enlightenment on the fundamentals.

There is plenty of material in this book to keep a class busy throughout the year. It is full of ideas—perhaps there are too many ideas all brought together without sufficient emphasis on those which are more important and far reach-

ing. The reviewer leaves the book with the impression that the student will be taught *how* to do a good many things in this subject, but he is even more impressed with the fact that he will not know *why* he is doing many of them. In short the interpretation which he will give to his results will often not be based upon a clear understanding of the principles upon which they are built. It is very desirable to use personal judgment and intuition in matters of the kind treated in this book, however, it is also very easy to go too far in this direction and to be over confident in the interpretation of the final results as obtained from a formula or process which is not understood. Nothing short of a sound knowledge of mathematics through the calculus will enable the student to understand the statistical tools described in this book and if he undertakes advanced statistical investigations he must go very much beyond the calculus before he will really have an intelligent understanding of what he is doing.

The literary style is good and the descriptions of the mere processes are straight forward and clear. The charts and tables are set up in desirable form and the book printed on excellent paper. In many respects the text is encyclopedic and will serve as a first class reference book of processes applied to various kinds of statistics. The author is to be congratulated on having made a useful addition to statistical literature.

JAMES W. GLOVER.

Advanced Calculus. By F. S. WOODS. New York, Ginn and Company, 1926. ix+397 pages. Price $4.60

The proper content of an advanced calculus course will long be a debatable question. There are many advocates who maintain that it should consist solely of further applications of differentiation and integration. There are some who believe it ought to be a repetition of the first course with added emphasis at certain points but little additional material (it should not be advanced at all, but *elementary* calculus should be made advanced). There are fewer who believe that a second course in calculus should deal primarily with the fundamentals in which the student should be grounded such as limits, continuity, convergence, summation, etc.

The present textbook by Woods proceeds in the middle of the road between the first and third points of view, and only incidently takes care of the second. Such fundamental concepts as continuity, uniform convergence, uniform continuity, dominant functions, existence theorems for implicit functions and for differential equations are not avoided, but given extensive and detailed consideration. There is no dodging of issues here, though by some strange turn, the fundamental definition of a limit of a function of a single variable in (δ, ϵ) terms is overlooked, though the corresponding one for continuity of a function at a point is given. The class of functions treated are in the main continuous

with continuous derivatives, except at a finite number of points. Frequently the function is analytic and the power series development is used effectively. The student is receiving important grounding which will stand him in good stead in his scientific future.

It is assumed by the author that the student has had a thorough training in formal integration and the use of a table of integrals and that he has had practice in setting up the simple and multiple integrals connected with length, area, volume, mass, center of mass, moment of inertia, pressure, etc., usually given as applications of integration in a first course. In the present text the properties of these definite integrals are discussed and proofs given in many instances. This extends also to Green's theorem in two and three dimensions and to Stokes' theorem, including the introduction of vector notation and definitions. The application of partial derivatives to the differential geometry of space curves and surfaces is beautifully done. This is the outstanding feature of the many excellent presentations of the entire book.

Four chapters are devoted to differential equations with special reference to the Legendre and Bessel equations. An entire chapter is given over to the very important and interesting properties of Bessel's functions, including their asymptotic expansions in a later chapter.

The Gamma and Beta functions are treated briefly. Analytic functions of a complex variable receive a chapter of twenty-eight pages. The book closes with a final chapter on Elliptic integrals and functions.

The sets of problems at the close of each chapter are numerous, averaging a little over forty for each of the sixteen chapters of the book. Very few if any of these problems are of the "turn the crank" type. Their content, for the most part, is such as to lead the student on to a further development of the subject.

It is an unpleasant task for the reviewer to call attention to a departure of the author at one point from the high standard set by the text as a whole. One of the most striking examples of the persistence of an erroneous tradition in mathematics is the incorrect form of Duhamel's theorem which continues to find a place in our text books. On page 22 of the present book under review, we find this theorem as follows:

"If the sum of n positive[1] infinitesimals has a limit as n increases indefinitely and each infinitesimal approaches zero, that limit is unaltered by replacing each infinitesimal by its principal part."

In 1903, W. F. Osgood showed that this theorem is false. However, the tradition will not down. For I have a copy of Duhamel's *Eléments de Calcul Infinitesimal* (the source of this theorem) before me and find, on page 35, the following

[1] The author in reality means non-negative.

THÉORÈME: *La limite de la somme de quantités posilives n'est pas changée, lorsqu'on remplace ces quantités par d'autres dont les rapport avec elles ont respectivement pour limite l'unité.* We find Gibson in an excellent text, *Elementary Treatise on the Calculus*, edition of 1919, has the identical statement of the incorrect form of the theorem in all its pristine beauty. Again this statement has made a recent appearance in Cohen's *Differential and Integral Calculus*, 1925, on page 275. Despite the fact that over twenty years ago Osgood gave a correct form for the theorem and others[1] have added further criticism, the false theorem appears in texts by eminent mathematicians.

Woods and others make a point of the non-negative character of the infinitesimals, whereas the "proof" and the examples by Woods on pages 22 and 23 show that it is the *boundedness* which is the essential element and not the likeness of sign of all the terms. It is this very quality which brings to grief Woods' immediate application of the theorem at the top of page 25. For example, if the arc under consideration in this application is a semi-circle with a horizontal diameter as base, all the required conditions are fulfilled, yet equations (4), (5) and (6) on page 25 become illusory. Except for certain definitions, sections 12 and 13 might very well be omitted from the present book without impairing its great value.

It is nevertheless true that the lack of an effective theorem of this kind is a distinct disadvantage. For example in section 59, page 140, on change of variable in a simple definite integral, various apologies are found necessary to justify the steps, such as "it is *usually* possible to split the interval (a, b) into regions in which the infinitesimals are always positive or always negative."

One may turn to W. F. Osgood's *Advanced Calculus* which appeared late in the year 1925 for comparison on this point. Despite the fact that Duhamel's theorem is not to be found in the index of Osgood's book, there are not less than eight applications of the theorem, (and at least a dozen other places where it might be applied) chiefly to problems in center of mass, attraction, surface integrals, polar coordinate integrals, spherical coordinate integrals, work, and change of variable in double and triple integrals. But Osgood does not introduce uniform convergence into his text, hence the application of his form of the theorem is ineffective as he himself states in several places.

The reviewer is so firmly convinced of the importance of a simple and correct form of the theorem (a simplified form of a theorem due to R. L. Moore) together with allied concepts, for the present and future development of the elementary parts of analysis, that he believes it ought to have a place in our elementary textbooks, certainly in advanced calculus books. Here, if you please, a tool can be placed in the hands of the student which in the beginning

[1] For complete references see my paper *A simple form of Duhamel's theorem and some new applications*, this MONTHLY, vol. 29 (1922), pp. 239–249.

may be used as a "crank" to obtain results, but which with greater mathematical maturity assumes a more and more definite meaning in the fundamental rôle which it plays in physical applications and in some advanced fields of analysis.

Other minor criticisms might be offered such as a bad and misleading figure on page 135 to represent a discontinuous function, and some inaccuracies of statement which will not be given detailed attention. But despite the vehemence of the reviewer's criticism of sections 12 and 13, he desires above all to point out that Woods has offered a distinct contribution to American mathematical textbooks in this excellent work on Advanced Calculus.

H. J. ETTLINGER.

An Introduction to Statistical Analysis. By G. G. CHAMBERS. New York, F. S. Crofts & Co., 1925. 254 pages.

According to the author this book represents the course in statistics which has been given for four semesters to first-year students in the University of Pennsylvania. It is supposed to be covered (apart from Chapter II or Chapter III) by the freshman in one semester of three hours per week. The applications are chiefly for students of biology and education. The book is extremely elementary, in fact too much so for university students. Practically the whole of the first chapter of twenty-two pages is employed in conveying to the student simple ideas of measurement, such as the length of a rod, or the weight and height of an individual. The final result obtained is that these measurements cannot be made exactly and that there will be a fluctuation from the true value which depends upon the smallness of the unit employed in taking the measures. The whole chapter could be condensed into two pages. This spirit of detail pervades the whole book and probably is a concession to the popular notion of the inability of most people to grasp elementary mathematical ideas and formulas. It would not be so surprising if the support of this notion came from a biologist or teacher of education but it is disappointing to find it coming from the pen of a mathematician. The second chapter deals with the most elementary examples on computation with approximations which one would expect to find in the text for a business college or in a "lightning calculator." The student is not supposed to know anything about logarithms so there is a chapter on this subject and another one on drawing simple graphs with much attention given to the rules proposed by the Joint Committee on Standards for Graphic Presentation. In chapter five a compilation of several pages of tables from original measurements is given and used as a basis for carrying out statistical problems developed in the remaining part of the book. Averages are discussed in chapter six and in chapter seven the ideas of dispersion and standard deviation are developed in much verbal detail. The chapter on cor-

relation is by far the best one in the book but the impression conveyed on reading it over—as well as the rest of the book for that matter—is that the author has left nothing for the student to think out for himself. Rules are given for each operation step by step just as one would direct a cook in preparing a dish. The final chapters on types of distributions and the theory of sampling would be more effective near the beginning of the book. The ideas are fundamental and should come to the attention of the student early in the course. The appendix contains a list of symbols and formulas given elsewhere in the book, also a five place table of common logarithms of numbers. The answers to every problem in the book are given in the appendix. The book is printed on good paper and the typing, style, tables and charts are well done.

The chief trouble with a book of this character is that after the student has completed it he is quite likely to believe that he knows much more about the subject than he really does. He ought to cover what is given in this book in four weeks instead of four months. The reviewer believes that statistics should not be approached in this manner in a university class in the department of mathematics. As a matter of fact the student should have freshman mathematics before he undertakes a course in elementary statistics which deals with measures of variation. JAMES W. GLOVER.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) the titles of papers in all mathematical journals published in the United States: (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Proceedings of the National Academy of Sciences, volume 12, no. 9, September 1926: "Postulates in the history of science" by G. A. Miller, 537–539; "On the zeros of functions associated with a linear system of the second order" by H. J. Ettlinger, 540–543; "Concerning irreducibly connected sets and irreducible continua" by H. M. Gehman, 544–547.

Science Progress, volume 20, no. 82, October 1926: "John Napier and the invention of logarithms" by G. A. Miller, 307–310.

Tokohu Mathematical Journal, volume 27, nos. 1, 2, July 1926: "Large primes having four consecutive quadratic residues" by A. A. Bennett, 53–57; "Applications of elliptic functions to the method of electrical images" by C. N. Wall, 176–188.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also

enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3232. Proposed by J. Rosenbaum, Milford, Connecticut.

Prove that in a tetrahedron, if three of the face angles at one vertex are equal, each less than or equal to a right angle, then any of the angles of the face opposite that vertex is less than the supplement of one of the three equal angles.

3233. Proposed by A. A. Bennett, Lehigh University.

Describe geometrically the continuous function represented by the Fourier series, $\sum \cos mt/m^2$, where m is restricted to positive integral values prime to 6. [This question arose in the theory of three-phase alternating currents].

3234. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The median through a given vertex of a triangle meets the Apollonian circle passing through this vertex and the circumcircle of the triangle in two points which, with the other two vertices of the triangle, determine a parallelogram.

3235. Proposed by Paul Capron, U. S. Naval Academy.

If the curtate cycloid, $x=a\phi-b\sin\phi$, $y=a-b\cos\phi$, $(b>a)$, has a node at which $x=0$ when $\phi=\alpha$, and if h_1, h_2, and $-h_3$ are its maximum ordinate, the ordinate of a node, and its minimum ordinate, show that the area of a loop is $\alpha(h_1h_3-ah_2)$.

If the loops are just large enough to be tangent externally, show that the area of a loop is about $39.2a^2=8.52ab=1.85b^2$.

SOLUTIONS

2662 [1918, 19; 1925, 316] Proposed by John Louke, New York City.

Assume that we have two piles of gold bars. The dimensions of the bars in the first pile are $2.643\times5.286\times10.573$ and the dimensions of the bars in the second pile are $2.13\times6.53\times10.573$. If possible, arrange the bars from the first pile and from the second pile so as to form perfect cubes, the bars from the piles to be taken separately or in combination.

SOLUTION BY HERBERT N. CARLETON, West Newbury, Mass.

Let the dimensions of the bars in the first pile be a/b, c/d, e/f, and those of the second, a'/b', c'/d', e'/f', where the letters represent positive integers. A cube may be formed by making a rectangular parallelopiped from the bars of the first pile having a square face, and joining it with another made from the second pile with an equal square face, such that their combined remaining dimension is the side of the square. Then

$$y\frac{c}{d}=z\frac{e}{f}=y'\frac{c'}{d'}=z'\frac{e'}{f'}=x\frac{a}{b}+x'\frac{a'}{b'},$$

where x, y, z, etc. are integers. A solution of these equations omitting the last is obviously

$$y=tdc'ee', \qquad z=tfcc'e',$$
$$y'=td'cee', \qquad z'=tf'cc'e.$$

Then

$$x\frac{a}{b}+x'\frac{a'}{b'}=tcc'ee'.$$

A solution of this equation is given by

$$x=qbcc'ee', \qquad x'=q'b'cc'ee', \qquad t=qa+q'a',$$

where q and q' are any positive integers. The side of the resulting cube is

$$(qa+q'a')cc'ee'.$$

NOTE BY THE EDITORS: For the numerical example of the problem, other solutions are possible. For example, using the first pile alone, a cube may be obtained with the edge 10.573×5286; using the second alone, with the edge $10.573\times30\times653\times71$; using both piles, with the edge $10.573\times5\times653\times5286\times71$.

3156 [3152; 1925, 481] Proposed by Otto Dunkel, Washington University.

From two fixed points, A and B of a given conic, two chords AC and BD are drawn intersecting on the fixed chord IJ of the same conic and determining another chord CD. Determine the envelope of the chord CD and a method for locating points on the envelope without the use of equations.

II. SOLUTION BY NATHAN ALTSHILLER-COURT, University of Oklahoma.

The four points A, B, C, D, determine a complete quadrangle inscribed in the given conic (O). Its diagonal points $E\equiv(AB, CD)$, $F\equiv(AC, BD)$, $G\equiv(AD, BC)$ determine a triangle which is self-polar with respect to (O), and since F lies on the line IJ, the polar EG of F will meet IJ in the conjugate F' of F with respect to (0) and will pass through the pole P of IJ. The pencil of rays $E-FGAD$ having E for its center is harmonic; hence the points $L\equiv(IJ, EAB)$ and $M\equiv(IJ, ECD)$ are harmonically separated by the points F, F'. Thus the line $CD\equiv ME$ joins the harmonic conjugate M, of the fixed point L with respect to the couple of conjugate points FF', to the projection E of F' from P upon the given line AB.

Starting with the point F of IJ we obtain the line ME. Now if we start with the point F', the roles of the points F, F' will be interchanged, while the points L, M will remain the same. Hence the line ME' joining M to the projection E' of F from P upon AB is another position of the variable line $CD\equiv ME$.

The lines PF, PF' are conjugate with respect to (0) because they project from the pole P of the line IJ two conjugate points F, F' of this line. As the point F varies on the fixed line IJ, the couple of conjugate points F, F' will describe an involution of points on IJ, and the point M will describe on IJ a range projective to this involution. Hence:

$$(M\cdots)\,\overline{\wedge}\,(FF'\cdots)\,\overline{\overline{\wedge}}\,P(FF',\cdots)\,\overline{\overline{\wedge}}\,(E'E,\cdots).$$

We have thus a projective one-to-two correspondence between the points of the two given lines, IJ and AB. Let L' be the conjugate of L with respect to the given conic (0) on the line IJ, and $H\equiv(PL', AB)$. The harmonic conjugate of the point L with respect to the couple LL' coincides with L; hence when the variable couple FF' of the line IJ coincides with the couple LL', the corresponding couple $E'E$ of AB coincides with the couple LH. Thus the common point L of the two bases IJ, AB is a united element of the two projective forms $(M\cdots)$ and $(E'E,\cdots)$. Consequently: *The two lines ME, ME' envelope a conic (M) tangent to the line AB.*

The lines ME, ME' determine on the conic (M) an involution of tangents of which the line IJ is the polar, and the point P is the pole. When the point F coincides with the point I, the conjugate F' of F with respect to (0) also coincides with I, hence the harmonic conjugate M of L with respect to F, F' will in its turn coincide with I. Thus from the point I of IJ only one tangent may be drawn to the conic (M). Similarly for the point J. Therefore the lines PI, PJ are the tangents to (M) at I, J. Thus: *The envelope (M) has a double contact with the given conic, the line IJ being the chord of contact.*

In the involution $(E'E,\cdots)$ on AB to the point L corresponds the point H, hence: *The conic (M) touches the line AB in the point H.*

Thus we have three tangents PI, PJ, AB to the conic (M) and their respective points of contact I, J, H. These elements are more than sufficient to generate the conic (M) either by points, or by lines, making use of Pascal's or Brianchon's theorems. For instance, the point of contact T of any tangent t to (M) may be found as follows: Let $X\equiv(t, PI)$, $Y\equiv(t, PJ)$, and $Z\equiv(IY, JX)$, then $T\equiv(t, PZ)$.

If the points I, J are conjugate imaginary we may use in this construction the line AB with its point of contact H and any other tangent which has been determined by means of the conic (0), the corresponding point of contact having been found by either Pascal's or Brianchon's theorem.

Also solved by C. T. BUNNEL, MICHAEL GOLDBERG, W. J. PATTERSON, and MABEL M. YOUNG.

3164 [3160; 1926, 47] Proposed by L. S. Dederick, University of British Columbia.

All values of a continuous variable between 0 and 10 are equally likely. What is the most probable number of values that must be selected at random before obtaining two that differ by more than 9?

Is the result the same if the phrase "most probable" is taken literally as if it has the common meaning as applied to errors, as likely as not to be exceeded.

Solution by the Proposer.

In order that any value shall be available as one member of a pair differing by more than 9, it must be less than 1 or greater than 9. The probability of this is $\frac{1}{5}$. The probability that out of n values exactly m shall be available is $C_m^n(\frac{1}{5})^m(\frac{4}{5})^{n-m}$. The probability that of these m available values exactly k shall be in the lower interval, 0 to 1, is $C_k^m(\frac{1}{2})^m$. If we arrange the available values in the order of magnitude of the *difference* between each one and the lower limit, 0 or 9, of the interval containing it, all orders are equally likely since all positions within the unit interval are equally likely. In this ordering, if no two values differ by more than 9, then all the values from the upper interval will precede all those from the lower. The probability of this is $k!(m-k)!/m!$ The probablity then that at least one pair shall differ by more than 9 is $1-[k!(m-k)!/m!]$ for any particular values of m and k. For an original list of n values the probability that the members of at least one pair shall differ by more than 9 is, therefore,

$$\sum_{m=2}^{n}\sum_{k=1}^{m-1} C_m^n\left(\frac{1}{5}\right)^m\left(\frac{4}{5}\right)^{n-m} C_k^m\left(\frac{1}{2}\right)^m\left[1-\frac{k!(m-k)!}{m!}\right]$$

$$= \sum_{m=2}^{n} C_m^n\left(\frac{1}{10}\right)^m\left(\frac{4}{5}\right)^{n-m}[2^m-1-m]$$

$$= \sum_{m=2}^{n} C_m^n\left(\frac{1}{5}\right)^m\left(\frac{4}{5}\right)^{n-m} - \sum_{m=2}^{n} C_m^n\left(\frac{1}{10}\right)^m\left(\frac{4}{5}\right)^{n-m} - \frac{n}{10}\sum_{m=2}^{n}\frac{(n-1)!}{(m-1)!(n-m)!}\left(\frac{1}{10}\right)^{m-1}\left(\frac{4}{5}\right)^{n-m}$$

$$= \left[1-\left(\frac{4}{5}\right)^n-\frac{n}{5}\left(\frac{4}{5}\right)^{n-1}\right]-\left[\left(\frac{9}{10}\right)^n-\left(\frac{4}{5}\right)^n-\frac{n}{10}\left(\frac{4}{5}\right)^{n-1}\right]-\frac{n}{10}\left[\left(\frac{9}{10}\right)^{n-1}-\left(\frac{4}{5}\right)^{n-1}\right]$$

$$= 1-\frac{n+9}{10}\left(\frac{9}{10}\right)^{n-1}.$$

If we call this $f(n)$ and set $f(n)=\frac{1}{2}$, we obtain the most probable value of n in the sense, the value as likely as not to be exceeded. By trial we find $f(16)=.4853$ and $f(17)=.5183$. Hence, the most probable number of values is between 16 and 17. If we select 17 values it is more likely than not that two of them will differ by more than 9; if we select 16 values it is less likely.

The most probable value of n, however, in the literal sense is quite different. Let

$$\phi(n) = f(n)-f(n-1) = \frac{n+8}{10}\left(\frac{9}{10}\right)^{n-2} - \frac{n+9}{10}\left(\frac{9}{10}\right)^{n-1}$$

$$= \left(\frac{9}{10}\right)^{n-2}\left(\frac{n+8}{10}-\frac{9n+81}{100}\right) = \frac{n-1}{100}\left(\frac{9}{10}\right)^{n-2}.$$

This is the probability that the nth value selected differs by more than 9 from one or more of the first $n-1$, but that none of these differs from another by more than 9, i. e., the nth choice is the first successful one. The function $\phi(n)$ continues to increase as long as $\phi(n+1)>\phi(n)$, that is

$$\frac{n}{100}\left(\frac{9}{10}\right)^{n-1} > \frac{n-1}{100}\left(\frac{9}{10}\right)^{n-2}$$

or $n<10$. We have then $\phi(10)=\phi(11)$ and this value is greater than any other value of $\phi(n)$. The probability, therefore, that the first successful choice is the 10th or 11th is greater than that of any preceding or following. In this sense 10 and 11 are the most probable values of n.

Also solved by Michael Goldberg.

3165 [3161; 1926, 47] Proposed by C. K. Robbins, Purdue University.

A non-degenerate conic and its tangents are inverted in $x^2+y^2=1$.

Show that the locus of the centers of the circles, which are the inverses of the tangents to the conic, is a conic.

SOLUTION BY W. L. AYRES, University of Pennsylvania.

Given the conic

$$x(a_{11}x + a_{12}y + a_{13}) + y(a_{21}x + a_{22}y + a_{23}) + (a_{31}x + a_{32}y + a_{33}) = 0, \tag{1}$$

where $a_{ij}=a_{ji}$, and where the determinant Δ of the coefficients is not zero, the equation of the tangent at x_1, y_1 is obtained by writing these coördinates for x, y within the parentheses. If in the resulting equation we now replace x, y by $x/(x^2+y^2)$, $y/(x^2+y^2)$, we obtain the equation of the inverse of the tangent, a circle. If $(\bar{x}, \bar{y})$ is the center then

$$\begin{aligned} -2\rho\bar{x} &= a_{11}x_1 + a_{12}y_1 + a_{13}, \\ -2\rho\bar{y} &= a_{21}x_1 + a_{22}y_1 + a_{23}, \\ \rho &= a_{31}x_1 + a_{32}y_1 + a_{33}. \end{aligned} \tag{2}$$

Since x_1, y_1 satisfy (1) we have at once

$$2x_1\bar{x} + 2y_1\bar{y} - 1 = 0. \tag{3}$$

From (2) we have

$$\begin{aligned} \Delta x_1 &= (-2D_{11}\bar{x} - 2D_{21}\bar{y} + D_{31})\rho, \\ \Delta y_1 &= (-2D_{12}\bar{x} - 2D_{22}\bar{y} + D_{32})\rho, \\ \Delta &= (-2D_{13}\bar{x} - 2D_{23}\bar{y} + D_{33})\rho, \end{aligned} \tag{4}$$

where D_{ij} is the cofactor of a_{ij} in Δ. Inserting (4) in (3) we obtain the equation of the desired locus, the conic,

$$2\bar{x}(-2D_{11}\bar{x} - 2D_{21}\bar{y} + D_{31}) + 2\bar{y}(-2D_{12}\bar{x} - 2D_{22}\bar{y} + D_{32}) - (-2D_{13}\bar{x} - 2D_{23}\bar{y} + D_{33}) = 0.$$

Since

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \Delta a_{33},$$

the locus is of the parabolic type if a_{33} is zero; elliptic, if Δ and a_{33} have the same sign; hyperbolic, if they have opposite signs.

Also solved by THEODORE BENNETT and MICHAEL GOLDBERG.

3166 [3162; 1926, 47] Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

Give an integral solution of $z^2=ay^2+x^2$

SOLUTION BY WILLIAM E. ROUTH, West Allis, Wis.

It is here assumed that a is an integer and that integral values of x, y and z satisfying $z^2=ay^2+x^2$ are desired. From this equation, we have

$$ay^2 = (z+x)(z-x);\ a^2y^2 = a(z+x)(z-x). \tag{1}$$

Since equation (1) must be satisfied by integers, it is evident that $z+x$ and $z-x$ must be integers of the form

$$z + x = mr^2t,\ z - x = ns^2t, \tag{2}$$

where m and n are integers such that $mn=a$, and r, s, and t are arbitrary integers. Then from (1) and (2),

$$y = \pm rst, \tag{3}$$

where the negative sign may be disregarded. Then

$$x = (mr^2 - ns^2)t/2,\quad z = (mr^2 + ns^2)t/2, \tag{4}$$

and the given equation is identically satisfied. Hence the integral solutions of that equation are given by (3) and (4), where t is even, or t is odd and mr^2+ns^2 is even.

Also solved by R. P. AGNEW, H. C. BRADLEY, P. A. CARIS, H. E. TREFETHEN and E. E. WHITFORD.

3171 [3167; 1926, 104]. Proposed by H. Betz, University of Missouri.

Consider a particle moving in a straight line in the plane of an ellipse, inside the ellipse in such a manner that whenever it strikes the boundary of the ellipse it is "reflected" just as a ray of light would be, striking a mirror. The particle will, therefore, travel indefinitely often back and forth across the ellipse.

Let its path be referred to as its orbit. Now if, initially, the orbit passes through one focus of the ellipse, it will, in accordance with an elementary property of the ellipse, pass through the other focus also, and so on, indefinitely. Show that the orbit will converge, in the limit, to the major axis of the ellipse.

SOLUTION BY THEODORE BENNETT, University of Illinois.

We assume the ellipse in the form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

the foci of which are $F(-ae,0)$, $F'=(ae,0)$. Let the particle start at $P=(\xi,\eta)$ on the ellipse, pass through F to Q, where it is reflected through F', striking the ellipse again at $P'=(\xi',\eta')$. Let us say that the operation T has moved the particle from P to P'. From geometrical considerations it is evident that the analytic expression of T is a bilinear relation between ξ and ξ' (but *not* between η and η'). We now find this relation. Eliminating y between the equations of the line PF and the ellipse we have a quadratic in x, one root of which is known to be ξ; the other root, the abscissa of Q, is found to be

$$X=-a\frac{\alpha\xi+a}{\xi+a\alpha}, \text{ where } \alpha=\frac{1+e^2}{2e}. \tag{1}$$

Similarly, starting from Q, we find the abscissa of P' to be

$$\xi'=a\frac{\alpha X-a}{X-a\alpha}. \tag{2}$$

Combining these results, we find the analytic expression of T to be

$$\xi'=a\frac{(1+\alpha^2)\xi+2a\alpha}{2\alpha\xi+a(1+\alpha^2)}. \tag{3}$$

Since the self-corresponding points of T are $\xi=a$ and $-a$, we know that (3) may be written

$$\frac{\xi'-a}{\xi'+a}=k\frac{\xi-a}{\xi+a}, \text{ or } \xi'=a\frac{(1+k)\xi+a(1-k)}{(1-k)\xi+a(1+k)}. \tag{4}$$

By comparing (3) and (4) we find that

$$k=\left(\frac{1-\alpha}{1+\alpha}\right)^2=\left(\frac{1-e}{1+e}\right)^4.$$

Now $\xi^{(n)}$, the abscissa of the point obtained from P by applying T n times, is obtained from (4) by replacing ξ' by $\xi^{(n)}$ and k by k^n. Since $0<e<1$, we have also $0<k<1$, whence $k^n\to 0$ as $n\to\infty$. Therefore, we see from (4) that $\xi^{(n)}\to a$ as $n\to\infty$, which proves the theorem.

NOTE BY OTTO DUNKEL, Washington University. This may also be proved geometrically by aid of the symmetry and convexity of the ellipse and of the focal property stated in the problem.

Also solved by W. J. PATTERSON, H. S. UHLER, and the PROPOSER.

3173 [3169; 1926, 104]. Proposed by C. C. Camp, University of Illinois.

Two parallel vertical walls stand upon horizontal ground. A ladder of length a has its foot at the bottom of the first wall and leans against the second. A ladder of length b has its foot at the bottom of the second wall and leans against the first. What must be the distance between the walls so that the ladders will cross at a height h? When is a solution possible?

SOLUTION BY W. J. PATTERSON, Western University, Can. and OTTO DUNKEL, Washington University.

Let x be the distance between the walls, and let the ladders of lengths a and b (both finite) meet their respective walls at the heights y and z. Let t be the distance of the foot of h from the foot of b. Then by similar triangles $t/x=h/z=(y-h)/y$. Hence

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{h}, \qquad h>0. \tag{1}$$

Since $y<a$ and $z<b$, we have as a necessary condition for a real solution

$$\frac{1}{a}+\frac{1}{b}<\frac{1}{h}. \tag{2}$$

It will be shown that (2) is a sufficient condition for a real solution.

If $a=b>2h_0$ then $y=z=2h_0$, and $x=(b^2-4h_0^2)^{1/2}$.

If $a>b$ and if a real solution is possible, then (1) shows that $2h$ is the harmonic mean of y and z, and hence $h<z<2h<y$. In order to determine x we have an additional equation

$$y^2-z^2=a^2-b^2=k^2, \tag{3}$$

and then from (1) and (3) we have

$$f(z)=z^2(z-2h)+k^2(z-h)^2=0. \tag{4}$$

From (4) it is obvious that $f(z)$ is positive if $z\leqq 0$ or if $z\leqq 2h$; moreover $f(h)=-h^4$. Hence there is at least one real root between 0 and h and at least one between h and $2h$. Now $f'(h)=-2h^3$; $f'(2h)=8h^3+2k^2h$, and since $f''(z)=12z(z-h)+2k^2$, it follows that $f'(z)$ has one and only one root in the interval $h,2h$. Thus $f(z)$ in this interval decreases from a negative value to a minimum and then increases and vanishes only once.

It must now be shown that this root z is less than b. We have after a slight change

$$f(b)=a^2(b-h)^2-h^2b^2, \tag{5}$$

while from (2) it follows that $a(b-h)>hb$. Hence $f(b)>0$, and since $b>h$, it follows that b is greater than this root z. Hence x is uniquely determined by $x=(b^2-z^2)^{1/2}$, and y is then found from (1). This pair of values of y and z satisfy (1) and (4) and hence must also satisfy (3). It now follows that

$$y^2=b^2-x^2+a^2-b^2=a^2-x^2.$$

This concludes the proof that this root and this one alone furnishes a solution.

We shall now consider the remaining roots of (4). If $2k^2-3h^2\geqq 0$, $f''(z)$ has a double root or imaginary roots; it then follows that $f(z)$ has a pair of imaginary roots. If $2k^2-3h^2<0$, $f''(z)$ has two real roots between 0 and h and at either one of these points $6f'(z)=(2z-h)f''(z)-4(3h^2-2k^2)z-10k^2h<0$. Since the maximum and minimum values of $f'(z)$ are negative, it must have two imaginary roots, and it then follows that $f(z)$ has a pair of imaginary roots. Thus the only other real root lies between 0 and h. The corresponding value of y for this smaller root is negative by (1). To this smaller root corresponds the case where the ladder a meets its wall below the horizontal, and the prolongations of the two ladders meet outside of the x interval. Here again a condition must be imposed upon a, b, h to obtain a real solution for this situation.

All of these results are easily obtained by an examination of the two hyperbolas in (1) and (3). It may be shown that if z is the greater root

$$2h - \frac{k^2}{8h} < z < 2h - \frac{k^2 h}{8h^2 + k^2}$$

The solution of the case $a = 40$, $b = 25$, $k = 15$, is given in the solution of 2836[1922, 181].

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

On Sunday, March 20th, 1927, will fall the two-hundredth anniversary of the death of Sir Isaac Newton. A meeting to celebrate this Bicentenary will be held on March 18, 19, under the auspices of the Yorkshire Branch of the Mathematical Association, at Grantham, the place associated with Newton's boyhood and with some of his most fundamental discoveries. There will be a scientific meeting with addresses on Newton's work by Sir J. J. Thomas, F. R. S. (Master of Trinity), Sir F. Dyson, F. R. S. (Astronomer Royal), Dr. J. H. Jeans (Sec. R. S.), Professor H. Lamb, F. R. S. After a pilgrimage to Woolsthorpe Manor House (Newton's birth place), there will be a visit to Stoke Rochford, where Mr. Christopher Turnor will speak on "Newton's Countryside."

Professor W. L. Hart has been appointed chairman of the Department of Mathematics in the College of Science, Literature, and the Arts at the University of Minnesota to succeed Professor W. H. Bussey who resigned the chairmanship so that he might devote more time to his duties as Assistant Dean and as Editor-in-Chief of the American Mathematical Monthly.

Dr. L. D. Ames has been appointed professor of mathematics at the University of Southern California.

Assistant Professor H. G. Harp of Wittenberg College is spending this academic year in graduate study at Ohio State University.

Dr. Harold R. Phalen, formerly associate professor of mathematics at the Armour Institute of Technology, has been appointed professor of mathematics at St. Stephen's College, Annandale-on-Hudson, New York.

Miss Una Rudd, instructor in mathematics in Simmons University is on leave this year, doing graduate work in the University of California.

Professor C. E. Horne, formerly Dean of the College of Agriculture at Mayaguez, Porto Rico, has been appointed professor of mathematics at the University of Porto Rico.

At Trinity College, Hartford, Connecticut, Mr. Howard T. Engstrom has been appointed instructor in mathematics. Assistant Professor Frederick J. Burkett is absent on leave for graduate study.

Mr. Alva M. Tuttle has been appointed instructor in mathematics at Heidelberg College, Tiffin, Ohio.

At the Case School of Applied Science, Cleveland, Ohio, Mr. Richard S. Burington has been appointed instructor in mathematics. Professor Max Morris is on leave of absence. Mr. E. M. Justin has returned to the department after a year of study at the University of Chicago.

Miss Esther Pearce, M. A. University of Michigan, has been appointed instructor in mathematics at Georgia Wesleyan College, Macon, Georgia.

Mr. Joseph H. Kusner has been appointed instructor in mathematics at the University of Florida.

At Ohio Wesleyan University, Mr. Sidney A. Rowland is professor of mathematics and head of the department. Earnest Clare Bower, formerly associate astronomer at the U. S. Naval Academy, has been appointed assistant professor of astronomy and mathematics.

At the University of Buffalo, Charles D. Gregory and C. Wallace Munshower have been appointed instructors in mathematics.

Professor Wilfred H. Sherk, of the University of Buffalo, has been elected President of the Association of the Teachers of Mathematics in the Middle States and Maryland.

Miss Lois Dicks (A. B. University of South Carolina), has been appointed instructor in mathematics at Coker College, to succeed Miss Ellen C. Stokes who resigned to accept a similar position in the New York State Teacher's College.

At Colgate University, Professor A. B. Stewart of the department of mathematics will retire from active teaching at the close of the present academic year. Professor A. W. Smith has leave of absence for the second semester of the present year.

At Kansas State Agricultural College, Mr. T. I. Porter, Mr. H. M. Stewart and Miss Irene Eldridge have been appointed instructors in mathematics; Mr. W. C. Janes, and Miss Thirza A. Mossman have been promoted from the rank of instructor to that of assistant professor; and Mr. C. F. Lewis, Mr. W. H. Lyons and Miss Emma Hyde from the rank of assistant professor to that of associate professor. Leave of absence has been granted to Professor A. E. White for study at the University of Iowa.

Mr. Dwight F. Gunder has been appointed instructor in mathematics at the Colorado Agricultural College to succeed Homer W. Craig who has accepted a fellowship at the University of Wisconsin.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 6-7, 1927.

Twelfth Annual Meeting, Nashville, Tenn., December, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 6-7.

INDIANA, De Pauw University, May 6-7.

IOWA, University of Iowa, 29, 30.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, May.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, May.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November.

ROCKY MOUNTAINS, Colorado College, April 22-23.

SOUTHEASTERN, March.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATION: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE MAY MEETING OF THE MINNESOTA SECTION

The regular spring meeting of the Minnesota Section was held at St. Olaf College, Northfield, Minnesota, on Saturday, May 22, 1926. Professor M. A. Nordgaard, St. Olaf College, the Chairman of the Section, presided.

The attendance was 45 at the dinner, 60 at the regular session, and included the following 21 members of the Association: W. O. Beal, A. Bogard, R. W. Brink, W. H. Bussey, J. M. Earl, C. H. Gingrich, Gladys Gibbens, W. L. Hart, D. Jackson, R. A. Johnson, D. N. Kingery, W. H. Kirchner, L. W. Moench, W. D. Morgan, M. A. Nordgaard, G. C. Priester, Sister Alice Irene, Ella Thorp, A. L. Underhill, Marion B. White, L. Winkelman.

The following officers were elected for the coming year: Chairman, INEZ RUNDSTROM, Gustavus Adolphus College, St. Peter, Minnesota; Secretary, A. L. UNDERHILL, University of Minnesota; an Executive Committee consisting of the Chairman, the Secretary and M. A. NORDGAARD, A. BOGARD, College of St. Teresa, Winona, Minnesota, and SISTER CLAUDETTE, St. Joseph College, St. Joseph, Minnesota.

A motion was passed expressing the appreciation of the Section for the hospitality of St. Olaf College.

The 1927 meeting will be held May 21 at Gustavus Adolphus College, St. Peter, Minnesota.

The following five papers were read:

1. "The correlation ratio," by Professor DUNHAM JACKSON, University of Minnesota.